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Stability Analysis of an SIR Epidemic Model with Proportional Mixing Incidence Rate and Treatment.

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ABSTRACT

In this paper, an SIR Epidemic model with proportional mixing incidence rate and treatment was considered. We determined the steady state of the model and the reproductive number R_0. Stability analysis was carried out and it was discovered that the disease free equilibrium of the model was locally asymptotically stable. Lyapunov function was employed to prove its global stability which was also asymptotically stable. Numerical simulation of the model carried out showed that as the treatment term increases the disease is fading off.

Keywords: Proportional Mixing Incidence Rate, Basic Reproductive Number, Equilibrium, Treatment, Stability

1. INTRODUCTION

Studies on mathematical modeling have been a popular area for the understanding, control and prediction of the spread of communicable diseases. This area of mathematics has been studied by many authors, of which most focus attention on formulating or evolving the existing incidence rate Li and Cui (2013). Several authors made use of the following incidence rate in an attempt to model communicable disease transmission processes, such as Bilinear incidence rate generally known as simple mass action βSI proposed by Kermac – Mckendrick (1927) where β denotes the transmission rate, while S *and* I denote Susceptible and Infected respectively , saturated incidence rate $g(I)S$ introduced by Capasso and Serio (1978), Ruan and Wang (2003) proposed $\frac{KI^I S}{1+\alpha I^h}$ where I and h are positive constants, $\frac{\beta SI}{1+\alpha_1 S+\alpha_2 I}$ Kaddar (2010).

Non- Linear incidence rate of the form $\frac{KI}{1+\alpha S^{P}+\beta I^{q}}$ proposed by Adebimpe et al (2015) where $p=q>1$ which is the modified form of incidence rate proposed by Pathark et al (2010) $\frac{KI}{1+\alpha S+\beta I}$, also Ankit Agrawal (2013) proposed an incidence rate of the form $\frac{KI}{1+\beta S}$.

In this paper, we considered the model proposed by Adebimpe et al (2015), by considering mixing proportional transmission rate $\frac{\beta SI}{\alpha(S+I)}$. We also looked into the local and global stabilities of the model. To ascertain the effect of transmission rate with the presence of treatment on the model, numerical analysis was carried out and the results were presented graphically.

2. THE BASIC MATHEMATICAL MODEL.

In this paper, *SIR* model proposed by Adebimpe et al (2015) was adopted and modified, by proposing a proportional mixing incidence rate. Local and Global Stability analysis of the disease free epidemic equilibrium were carried out.

The existing model from Adebimpe et al (2015) was given as:

$$\left.\begin{aligned}\frac{dS}{dt}&=b-dS-\frac{KI}{1+\alpha S^{p}+\beta I^{q}}+\gamma R\\ \frac{dI}{dt}&=\frac{KI}{1+\alpha S^{p}+\beta I^{q}}-(d+\mu)I-\tau I\\ \frac{dR}{dt}&=\mu I-(d+\gamma)R+\tau I\end{aligned}\right\} \tag{1}$$

Modified Model.

The modified model is as follows:

$$\left.\begin{aligned}\frac{dS}{dt}&=b-dS-\frac{\beta SI}{\alpha S+\alpha I}+\gamma R\\ \frac{dI}{dt}&=\frac{\beta SI}{\alpha S+\alpha I}-(d+\mu)I-\tau I\\ \frac{dR}{dt}&=\mu I-(d+\gamma)R+\tau I\end{aligned}\right\} \tag{2}$$

With $S(t), I(t)$ *and* $R(t) \neq (0,0,0)$

b is the population recruitment rate, μ is the recovery rate, γ is the rate at which recovered individuals loose immunity and return to Susceptible, d is the natural death rate, and α is the parameter which measures the psychological or sociological effect, while S, I, R denoted Susceptible, Infected and Recovery class respectively.

3. STEADY STATE AND LOCAL STABILITY OF THE EQUILIBRIUM POINTS.

Local stability of the disease – free equilibrium and endemic equilibrium of the system of equation (2) will be discussed in this section.

The equilibrium of eqn. (2) must satisfy the following condition:

$$\left.\begin{aligned} \frac{dS}{dt} &= b - dS - \frac{\beta SI}{\alpha S + \alpha I} + \gamma R = 0 \\ \frac{dI}{dt} &= \frac{\beta SI}{\alpha S + \alpha I} - (d + \mu)I - \tau I = 0 \\ \frac{dR}{dt} &= \mu I - (d + \gamma)R + \tau I = 0 \end{aligned}\right\} \tag{3}$$

In the absence of disease infection in the population, $I = 0$. Solving eqn (3) gives disease – free equilibrium as:

$$E_0 = (S, I, R) = \left(\frac{b}{d}, 0, 0\right) \tag{4}$$

Also, in the presence of disease infection in the population, $I \neq 0$. Solving (3) admits a unique solution (Endemic equilibrium).

$$E_* = (S_*, I_*, R_*) \tag{5}$$

Where;

$$S_* = \frac{b(\alpha(d + \gamma)(d + \mu + \tau))}{d(\beta(d + \gamma + \tau) + \beta\gamma - \alpha d(\mu + \tau) - \alpha(\mu^2 + 2\mu\tau + \tau^2))} \tag{6}$$

$$I_* = \frac{(\beta - d - \mu - \tau)(d + \gamma)b}{d\beta(d + \gamma + \mu + \tau) - d(\mu + \tau) + \mu^2 - 2\mu\tau - \tau^2} \tag{7}$$

$$R_* = \frac{(\beta - d - \mu - \tau)(d + \tau)b}{d\beta(d + \gamma + \mu + \tau) - d(\mu + \tau) + \mu^2 - 2\mu\tau - \tau^2} \tag{8}$$

3.1 Local Stability of Disease – Free Equilibrium

Preposition 1.

If $R_0 < 1$, then the disease free equilibrium E_0 is locally asymptotically stable.

Proof.

Considering Linearization method, the resulting characteristic equation of system (2) is $|A - \lambda I| = 0$

$$p(\lambda) = \begin{vmatrix} -(d+\lambda) & -\beta & \gamma \\ 0 & \frac{\beta}{\alpha} - d - \mu - \tau - \lambda & 0 \\ 0 & \mu + \tau & -(d+\gamma) - \lambda \end{vmatrix} \tag{9}$$

Solving the determinant of the characteristic equation (9) we have;

$$-(d+\lambda)\left(\left(\frac{\beta}{\alpha} - d - \mu - \tau - \lambda\right)\left(-(d+\gamma) - \lambda\right)\right) = 0 \tag{10}$$

Solving (10), it is obviously shown that it has roots;

$$\lambda_1 = -d, \qquad \lambda_2 = \frac{\beta}{\alpha} - d - \mu - \tau - \lambda \qquad \text{and} \qquad \lambda_3 = -(d+\gamma) \tag{11}$$

Let $R_0 = \dfrac{\beta}{\alpha(d+\mu+\tau)}$ (12)

Hence , if $R_0 < 1$, the disease free equilibrium is locally asymptotically stable and otherwise if $R_0 > 1$.

3.2. Local Stability of Endemic Equilibrium.

Preposition 2.

The endemic equilibrium E_* of the model (2) is locally asymptotically stable if $R_0 > 1$.

Proof.

Let the endemic equilibrium $E_* = (S_*, I_*, R_*)$, using linearization method by setting

$S - S_* = x$, $I - I_* = y$, $R - R_* = z$

Then the resulting Jacobian matrix of system of equation (2) is

$$J(E_*)=\begin{vmatrix} -d-\dfrac{\beta I_*}{\alpha S_*+\alpha I_*}+\dfrac{\beta S_* I_* \alpha}{(\alpha S_*+\alpha I_*)^2} & -\dfrac{\beta S_*}{\alpha S_*+\alpha I_*}+\dfrac{\beta S_* I_* \alpha}{(\alpha S_*+\alpha I_*)^2} & \gamma \\ \dfrac{\beta I_*}{\alpha S_*+\alpha I_*}-\dfrac{\beta S_* I_* \alpha}{(\alpha S_*+\alpha I_*)^2} & \dfrac{\beta S_*}{\alpha S_*+\alpha I_*}-\dfrac{\beta S_* I_* \alpha}{(\alpha S_*+\alpha I_*)^2}-d-\mu-\tau & 0 \\ 0 & \mu+\tau & -d-\gamma \end{vmatrix} \tag{11}$$

Hence, the characteristic equation is

$$P(\lambda)=\begin{vmatrix} -d-\dfrac{\beta I_*}{\alpha S_*+\alpha I_*}+\dfrac{\beta S_* I_* \alpha}{(\alpha S_*+\alpha I_*)^2}-\lambda & -\dfrac{\beta S_*}{\alpha S_*+\alpha I_*}+\dfrac{\beta S_* I_* \alpha}{(\alpha S_*+\alpha I_*)^2} & \gamma \\ \dfrac{\beta I_*}{\alpha S_*+\alpha I_*}-\dfrac{\beta S_* I_* \alpha}{(\alpha S_*+\alpha I_*)^2} & \dfrac{\beta S_*}{\alpha S_*+\alpha I_*}-\dfrac{\beta S_* I_* \alpha}{(\alpha S_*+\alpha I_*)^2}-d-\mu-\tau-\lambda & 0 \\ 0 & \mu+\tau & -d-\gamma-\lambda \end{vmatrix}=0 \tag{12}$$

Solving the determinant of the characteristic equation (12) we have;

$$\lambda^3+\lambda^2 a_1+\lambda a_2+a_3 \tag{13}$$

Where

$$a_1=\frac{\beta I_*}{\alpha S_*+\alpha I_*}-\frac{\beta S_*}{\alpha S_*+\alpha I_*}-d-\mu-\tau-\gamma \tag{14}$$

$$a_2=d+\gamma-\left(d+\frac{\beta I_* \alpha}{\alpha S_*+\alpha I_*}-\frac{\beta S_* I_* \alpha}{(\alpha S_*+\alpha I_*)^2}\right)\left(\frac{\beta S_* \alpha}{\alpha S_*+\alpha I_*}-\frac{\beta S_* I_* \alpha}{(\alpha S_*+\alpha I_*)^2}-d-\mu-\gamma\right) \tag{15}$$

$$a_3=\left(d+\frac{\beta I_* \alpha}{\alpha S_*+\alpha I_*}-\frac{\beta S_* I_* \alpha}{(\alpha S_*+\alpha I_*)^2}\right)\left(\frac{\beta S_* \alpha}{\alpha S_*+\alpha I_*}-\frac{\beta S_* I_* \alpha}{(\alpha S_*+\alpha I_*)^2}-d-\mu-\gamma\right)(d-\gamma) \tag{16}$$

Considering Routh – Hurwitz criteria which stated that all characteristics root must have negative real parts. Since $a_2 a_1-a_3>0$ then the endemic equilibrium is locally asymptotically stable.

3.3. Global Stability of the Disease Free Equilibrium.

Considering the Lyapunov function, we have

$$L(I) = I \tag{17}$$

$$L^{'} = I^{'} = \frac{\beta SI}{\alpha(S+I)} - (d+\mu+\sigma)I \tag{18}$$

$$L^{'} = (d+\mu+\sigma)\left(\frac{\beta SI}{\alpha(d+\mu+\sigma)(S+I)} - 1\right)I \tag{19}$$

At disease free equilibrium, we obtained

$$L^{'} = (d+\mu+\sigma)(R_0 - 1)I \tag{20}$$

Since $I > 0$, if $R_0 \leq 1$ then $L^{'} \leq 0$

Hence, the disease – free equilibrium is globally asymptotically stable.

4. RESULT AND DISCUSSION.

To investigate the behavior of system (2), the numerical solution of system (2) was obtained using method of Runge - Kutta of order 4, varying some key parameters of the model and other parameters kept constant. The results were presented graphically as follows;

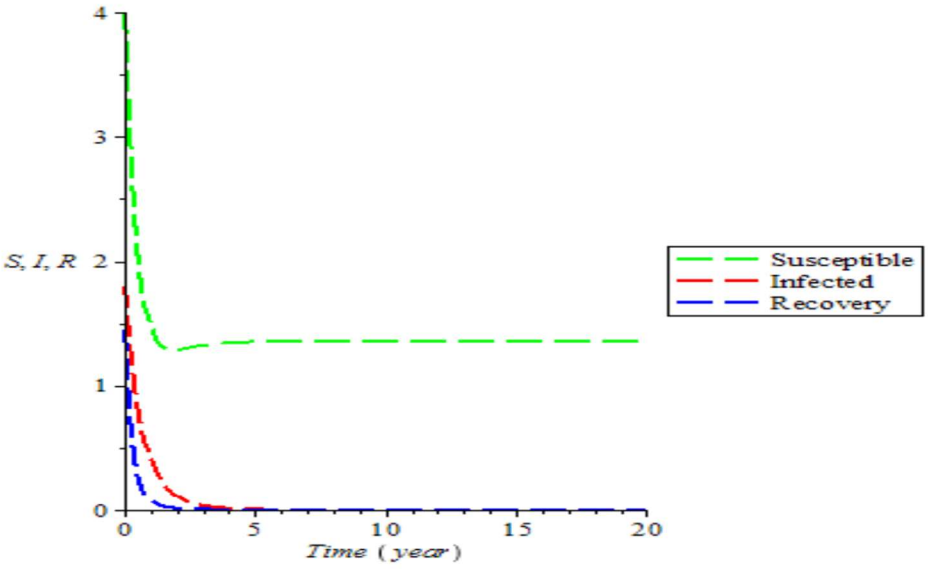

Fig.1: Graph of Susceptible, Infected and Recovery when
$\beta = 4.7,\ \mu = 0.19, \gamma = 1.5,\ d = 2.29, b = 3.1, \tau = 0.1, \alpha = 3.1$

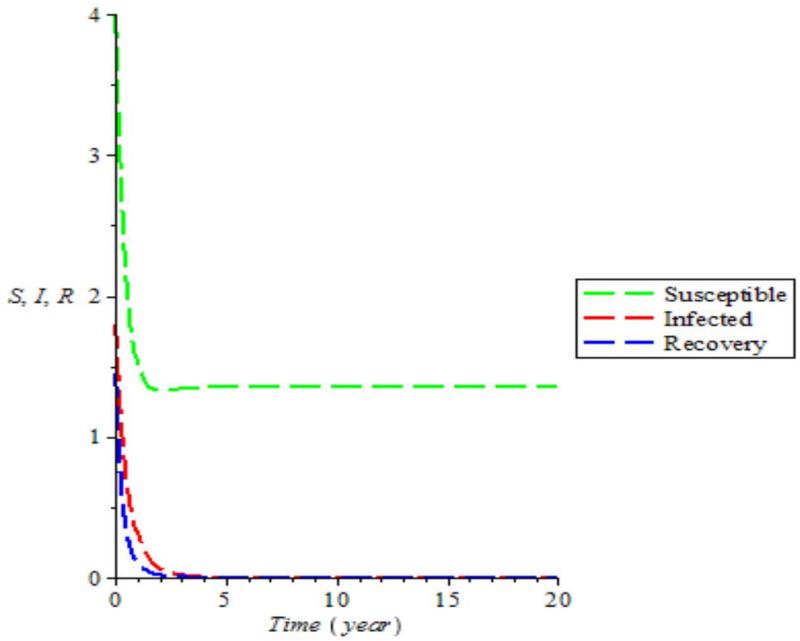

Fig.2: Graph of Susceptible, Infected and Recovery when
$\beta = 4.7,\ \mu = 0.19, \gamma = 1.5,\ d = 2.29, b = 3.1, \tau = 0.4, \alpha = 3.1$

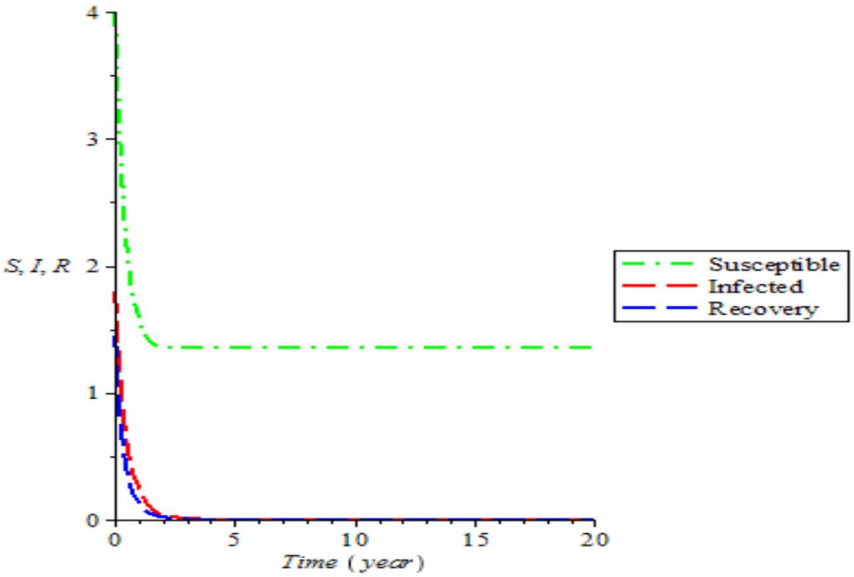

Fig.3: Graph of Susceptible, Infected and Recovery when
$\beta = 4.7,\ \mu = 0.19, \gamma = 1.5,\ d = 2.29, b = 3.1, \tau = 0.7, \alpha = 3.1$

In fig. 1, it was clearly shown that the disease Infected and Recovery class reduces, while the reproductive number $R_0 = 0.5876$ when $\beta = 4.7$ and the treatment term $\tau = 0.1$. In fig. 2, we observed that the disease Infected and Recovery classes reduced further, when the treatment term was increased from $\tau = 0.1$ *to* $\tau = 0.4$ at constant value of $\beta = 4.7$. The reproductive number also reduced from $R_0 = 0.5876$ to $R_0 = 0.5267$, the infected and Recovery classes later approached the steady state. It can be seen in fig. 3, that when the treatment term was further increased from $\tau = 0.14$ *to* $\tau = 0.7$ at constant value of $\beta = 4.7$, the disease Infected and Recovery classes were reduced to equilibrium point as time tended to infinity. It was also noticed that the reproductive number R_0 further decreased to $R_0 = 0.4768$.

5. CONCLUSION

In this paper, we investigated an SIR model with proportional mixing incidence rate. R_0 was calculated and it was observed that if $R_0 < 1$ the disease free equibilirium was locally asymptotically stable and globally stable asymptotically if $R_0 > 1$. The behavioral analysis of the model was carried out using numerical simulation to investigate the effect of treatment term on the model. Using some parameters from the reviewed literature, it was clearly shown that the treatment term had a significant effect in the progression of the disease. Increasing the treatment term brought the reduction on the infected and the disease gradually fades off.

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