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## On the Discrete Logarithm Problem of the Finite Groups of Rhotrices over $Z_p$

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### ABSTRACT

The Discrete Logarithm Problem (DLP) of finite groups of rhotrices over  $Z_p$  was explored. The representation and manipulation of these groups were provided in order to lay a solid foundation for the study. Numerical examples were also given to further substantiate the DLP.

**Keywords:** Group; Finite group; Cyclic group; Cryptography; Discrete logarithm problem.

### 1. INTRODUCTION

Clearly, the security of the Discrete Logarithm Problem (DLP) depends heavily on the group (finite)  $G$  under consideration. The more difficult the DLP the more secured it is. The group  $G = F_{p,l}^*$  for a chosen prime  $p$  and exponent  $l$  or the group of points of a suitably chosen elliptic curve over a finite field can be used [1]. Hence, if the DLP is made easier it becomes more insecure. In this paper, emphases were put on the suitable choice of groups, description on how group elements are stored and manipulated. It was hoped that this effort will further enrich the pedagogical aspect of cryptography.

### 2. REVIEW OF RELATED LITERATURE

A rhotrix is an object:

$$A = \left\langle \begin{matrix} a & \\ b & c & d \\ e & & \end{matrix} \right\rangle$$

belonging to the set  $R$  where  $a, b, c, d, e \in \mathfrak{R}$  [2]. Rhotrices is the plural of rhotrix and the central entry is

called heart. If  $A$  and  $B = \left\langle \begin{matrix} g & \\ h & j & k \\ l & & \end{matrix} \right\rangle$  are two rhotrices,



It was given that

$$0 = \begin{pmatrix} & & 0_1 & & & & \\ & & 0_2 & 0_3 & 0_4 & & \\ & \dots & \dots & \dots & \dots & \dots & \\ 0_\alpha & \dots & \dots & 0_\beta & \dots & \dots & 0_\pi \\ & \dots & \dots & \dots & \dots & \dots & \\ & & 0_{t-3} & 0_{t-2} & 0_{t-1} & & \\ & & & 0_t & & & \end{pmatrix} \text{ is the additive identity of } M \text{ and its}$$

multiplicative identity is

$$I = \begin{pmatrix} & & 0_1 & & & & \\ & & 0_2 & 0_3 & 0_4 & & \\ & \dots & \dots & \dots & \dots & \dots & \\ 0_\alpha & \dots & \dots & 1_\beta & \dots & \dots & 0_\pi \\ & \dots & \dots & \dots & \dots & \dots & \\ & & 0_{t-3} & 0_{t-2} & 0_{t-1} & & \\ & & & 0_t & & & \end{pmatrix}$$

and if  $n = p$  where  $p$  is a prime, then the multiplicative inverse of

$$A = \begin{pmatrix} & & a_1 & & & & \\ & & a_2 & a_3 & a_4 & & \\ & \dots & \dots & \dots & \dots & \dots & \\ a_\alpha & \dots & \dots & a_\beta & \dots & \dots & a_\pi \\ & \dots & \dots & \dots & \dots & \dots & \\ & & a_{t-3} & a_{t-2} & a_{t-1} & & \\ & & & a_t & & & \end{pmatrix} \in M$$

will be

$$B = \begin{pmatrix} & & b_1 & & & & \\ & & b_2 & b_3 & b_4 & & \\ & \dots & \dots & \dots & \dots & \dots & \\ b_\alpha & \dots & \dots & b_\beta & \dots & \dots & b_\pi \\ & \dots & \dots & \dots & \dots & \dots & \\ & & b_{t-3} & b_{t-2} & b_{t-1} & & \\ & & & b_t & & & \end{pmatrix} \in M$$

Such that for  $i = \beta$

$$a_\beta d_\beta = 1 \pmod p$$

and for each  $i = 1, 2, \dots, t$  and  $i \neq \beta$

$$a_i d_\beta + a_\beta d_i = 0 \pmod p$$

The following definitions are presented according to [5]: if  $G$  is a group and there exists an element  $a$  in  $G$  such that  $G = \{a^m : m \in \mathbb{Z}\}$ , then  $G$  is called a cyclic group and the element  $a$  is said to be the generator of  $G$ . Also recall that, a reflexive, symmetric and transitive relation  $R$  in the set  $X$  is called an equivalence relation. Moreover, if  $\sim$  is an equivalence relation on a set  $X$ , for  $a \in X$ , the equivalence class of  $a$  is the set of all elements  $b \in X$  such that  $a \sim b$  and is denoted by  $[a]$ .

### FINITE GROUPS OF RHOTRICES OVER $Z_p$

By [6], let

$$x = \begin{pmatrix} & & x_1 & & & & & & & & \\ & & x_2 & x_3 & x_4 & & & & & & \\ x_\alpha & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & x_\pi \\ & & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \\ & & x_{t-3} & x_{t-2} & x_{t-1} & & & & & & \\ & & & x_t & & & & & & & \end{pmatrix}$$

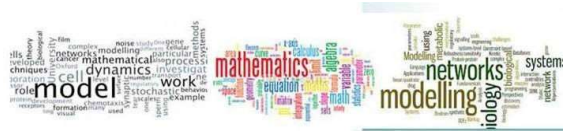
$x_i \in Z_p, i = 1, 2, \dots, t$  for some prime  $p, x_i \neq 0$  for each  $i, x_\beta \geq 2$  such that either all the  $x_i$ 's are equal or they are all equal with the exception  $x_\beta$ . In this paper, the case where  $x_i \in Z_p, i = 1, 2, \dots, t$  for some prime  $p, x_i \neq 0$  for each  $i, x_\beta \geq 2$  such that the  $x_i$ 's may or may not be equal to each other was studied.

- Let  $G$  be the set of order  $p(p - 1)$ , generated by  $x$  under the multiplication ( $\circ$ ) of rhotrices modulo  $p$ , then  $G$  is a group. For example, suppose  $p = 5, t = 5$ , then the order of  $G$  is  $5 \times 4 = 20$ .

Meanwhile, Let  $\mu = \begin{pmatrix} 3 \\ 4 & 3 & 2 \\ 1 \end{pmatrix} \in G$ , be the generator of  $G$ , then

$$\mu^1 = \begin{pmatrix} 3 \\ 4 & 3 & 2 \\ 1 \end{pmatrix}$$

$$\mu^2 = \begin{pmatrix} 3 \\ 4 & 4 & 2 \\ 1 \end{pmatrix}$$



$$\mu^3 = \begin{pmatrix} 1 \\ 3 & 2 & 4 \\ 2 \end{pmatrix}$$

$$\mu^4 = \begin{pmatrix} 4 \\ 2 & 1 & 1 \\ 3 \end{pmatrix}$$

$$\mu^5 = \begin{pmatrix} 0 \\ 0 & 3 & 0 \\ 0 \end{pmatrix}$$

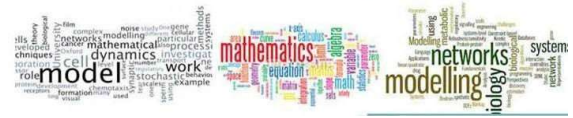
$$\mu^6 = \begin{pmatrix} 4 \\ 2 & 4 & 1 \\ 3 \end{pmatrix}$$

$$\mu^7 = \begin{pmatrix} 4 \\ 2 & 2 & 1 \\ 3 \end{pmatrix}$$

$$\mu^8 = \begin{pmatrix} 3 \\ 4 & 1 & 2 \\ 1 \end{pmatrix}$$

$$\mu^9 = \begin{pmatrix} 2 \\ 1 & 3 & 3 \\ 4 \end{pmatrix}$$

$$\mu^{10} = \begin{pmatrix} 0 \\ 0 & 4 & 0 \\ 0 \end{pmatrix}$$



$$\mu^{11} = \left\langle \begin{array}{c} 2 \\ 1 \ 2 \ 3 \\ 4 \end{array} \right\rangle$$

$$\mu^{12} = \left\langle \begin{array}{c} 2 \\ 1 \ 1 \ 3 \\ 4 \end{array} \right\rangle$$

$$\mu^{13} = \left\langle \begin{array}{c} 4 \\ 2 \ 3 \ 1 \\ 3 \end{array} \right\rangle$$

$$\mu^{14} = \left\langle \begin{array}{c} 1 \\ 3 \ 4 \ 4 \\ 2 \end{array} \right\rangle$$

$$\mu^{15} = \left\langle \begin{array}{c} 0 \\ 0 \ 2 \ 0 \\ 0 \end{array} \right\rangle$$

$$\mu^{16} = \left\langle \begin{array}{c} 1 \\ 3 \ 1 \ 4 \\ 2 \end{array} \right\rangle$$

$$\mu^{17} = \left\langle \begin{array}{c} 1 \\ 3 \ 3 \ 4 \\ 2 \end{array} \right\rangle$$

$$\mu^{18} = \left\langle \begin{array}{c} 2 \\ 1 \ 4 \ 3 \\ 4 \end{array} \right\rangle$$



$$\mu^{19} = \begin{pmatrix} 3 \\ 4 & 2 & 2 \\ 1 \end{pmatrix}$$

$$\mu^{20} = \begin{pmatrix} 0 \\ 0 & 1 & 0 \\ 0 \end{pmatrix}$$

mod  $p$  are the 20 elements of the group  $G$ .

**3. THE DISCRETE LOGARITHM PROBLEM (DLP)**

**Theorem:**

Let  $G$  be as defined in section 2 above. Let  $g = \begin{pmatrix} & & g_1 & & & & \\ & & g_2 & g_3 & g_4 & & \\ g_\alpha & \dots & \dots & \dots & \dots & \dots & \\ & & \dots & \dots & g_\beta & \dots & \dots & g_\pi \\ & & \dots & \dots & \dots & \dots & \dots & \\ & & g_{t-3} & g_{t-2} & g_{t-1} & & \\ & & & g_t & & & \end{pmatrix}$  be a generator of  $G$ : If an

element  $y = \begin{pmatrix} & & y_1 & & & & \\ & & y_2 & y_3 & y_4 & & \\ y_\alpha & \dots & \dots & \dots & \dots & \dots & \\ & & \dots & \dots & y_\beta & \dots & \dots & y_\pi \\ & & \dots & \dots & \dots & \dots & \dots & \\ & & y_{t-3} & y_{t-2} & y_{t-1} & & \\ & & & y_t & & & \end{pmatrix}$  in  $G$  is chosen, find a positive integer  $n$  such that  $g^n = y$ .

**Proof:**

By the rhotrix exponent rule provided by [7] and the hypothesis above,

$$g^n = \begin{pmatrix} & & g_1 & & & & \\ & & g_2 & g_3 & g_4 & & \\ g_\alpha & \dots & \dots & \dots & \dots & \dots & \\ & & \dots & \dots & g_\beta & \dots & \dots & g_\pi \\ & & \dots & \dots & \dots & \dots & \dots & \\ & & g_{t-3} & g_{t-2} & g_{t-1} & & \\ & & & g_t & & & \end{pmatrix}^n$$

$$\begin{aligned}
 &= g_{\beta}^{n-1} \begin{pmatrix} ng_1 & & & & \\ & ng_2 & ng_3 & ng_4 & \\ & \dots & \dots & \dots & \dots \\ ng_{\alpha} & \dots & \dots & ng_{\beta} & \dots & \dots & ng_{\pi} \\ & \dots & \dots & \dots & \dots & \dots & \\ & & ng_{t-3} & ng_{t-2} & ng_{t-1} & & \\ & & & ng_t & & & \\ & & & y_1 & & & \\ & & & y_2 & y_3 & y_4 & \\ y_{\alpha} & \dots & \dots & y_{\beta} & \dots & \dots & y_{\pi} \\ & \dots & \dots & \dots & \dots & \dots & \\ & & y_{t-3} & y_{t-2} & y_{t-1} & & \\ & & & y_t & & & \end{pmatrix} \\
 &= \begin{pmatrix} & & & & & & \\ & & & & & & \\ & \dots & \dots & \dots & \dots & \dots & \\ y_{\alpha} & \dots & \dots & y_{\beta} & \dots & \dots & y_{\pi} \\ & \dots & \dots & \dots & \dots & \dots & \\ & & y_{t-3} & y_{t-2} & y_{t-1} & & \\ & & & y_t & & & \end{pmatrix}
 \end{aligned}$$

This implies

$$g_{\beta}^{n-1} ng_{\beta} = g_{\beta}^n = y_{\beta} \quad (1)$$

$$g_{\beta}^{n-1} ng_i = y_i \quad (2)$$

Dividing (2) by (1), we have

$$\frac{g_{\beta}^{n-1} ng_i}{g_{\beta}^n} = \frac{y_i}{y_{\beta}}$$

Implying that

$$\frac{ng_i}{g_{\beta}} = \frac{y_i}{y_{\beta}}$$

Therefore,

$$n = \frac{y_i g_{\beta}}{y_{\beta} g_i}$$

### 3.1 Implementation

Let Alice and Bob agree on an arbitrary finite group  $G$  of rhotrices and its generator  $g$  as described in section 2 above. To secure an exchange of information that is going to take place between both of them:

- 1) Alice chooses a random key  $y$  in  $G$  which can be expressed as a power of  $g$  in  $G$  and sends the message under this key to Bob
- 2) Bob searches for a positive integer  $n$  such that  $y = g^n$  to recover the message.



### 3.2 Test

1) Suppose Alice and Bob agree on the group  $G$  of section 2 above. If Alice sends the message under the key

$$y = \begin{pmatrix} 4 \\ 2 & 4 & 1 \\ 3 \end{pmatrix}$$

which is a power of  $g = \begin{pmatrix} 4 & 3 & 2 \\ 1 \end{pmatrix}$ , then Bob searches for

$n = \frac{y_i g_i}{y_j g_j}$ , which means  $n = \frac{4 \times 3}{3 \times 4} = \frac{2 \times 3}{4 \times 4} = \frac{1 \times 3}{2 \times 4} = \frac{3 \times 3}{1 \times 4} = 1 \pmod{5}$ . Therefore,  $n$  falls in the equivalence class of  $1 \pmod{5}$  and since  $n$  is up to 20, then  $n = 1$  or  $6$  or  $11$  or  $16$ , but by testing, clearly  $n$  is not  $1$  or  $11$  or  $16$  and therefore  $n = 6$ .

2) If  $g = \begin{pmatrix} 3 \\ 4 & 3 & 2 \\ 1 \end{pmatrix}$  and  $y = \begin{pmatrix} 3 \\ 4 & 1 & 2 \\ 1 \end{pmatrix}$ , then  $n = \frac{3 \times 3}{3 \times 1} = \frac{4 \times 3}{4 \times 1} = \frac{1 \times 3}{1} = \frac{4}{3} = \frac{2}{4} = \frac{1}{2} = \frac{3}{1} = 3 \pmod{5}$

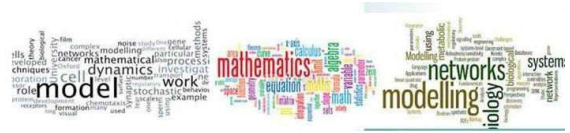
Since for instance,  $\frac{4}{3}$  is the same as multiplying 4 by the multiplicative inverse of  $3 \pmod{5}$

i.e.  $\frac{4}{3} = 4 \times 2 = 8 \equiv 3 \pmod{5}$ . Similarly for  $\frac{2}{4}$  and  $\frac{1}{2}$ .

Now, since  $n = 3$ , it implies that  $n$  falls in the equivalence class of  $3 \pmod{5}$  i.e.  $n = 3$  or  $8$  or  $13$  or  $18$ , but

### 4. CONCLUSION

This paper displayed the implementation of Discrete Logarithm Problem of certain finite groups of rhotrices. The implementation was tested in order to show how it works practically. Needed definitions and results were brought forward under relevant sections to either prepare or refresh the reader to have a clear understanding of the paper. The paper succeeded in showing a particular area of application of rhotrices.



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