Article Citation Format
Alabi, M.O. & Babatunde, O.Z. (2019): A Six Step Block Grid Point Collocation Method for Direct Solution of Second Order Ordinary Differential Equations Using Chebyshev Polynomials as Basis Function. Journal of Advances in Mathematical & Computational Sciences Vol. 7, No. 2. Pp 9-18

Article Progress Time Stamps

Article Type: Research Article
Manuscript Received 23rd April. 2019
Review Type: Blind
Final Acceptance: 11th June, 2019
Article DOI:
dx.doi.org/10.22624/AIMS/MATHS/V7N2P2

A Six Step Block Grid Point Collocation Method for Direct Solution of Second Order Ordinary Differential Equations Using Chebyshev Polynomials as Basis Function.

Alabi. M.O. PhD
Department of Physical Sciences
College of Natural and Applied Sciences
Chrisland University
Abeokuta, Ogun State, Nigeria
E-mail: moolaalabi@yahoo.com

Babatunde, O.Z.
School of Science,
Federal College of Education (Special)
Oyo, Oyo State, Nigeria

ABSTRACT

The application of linear multistep methods in solving differential equations cannot be over-emphasized. Many authors have come up with diverse methods of solving differential equations in which some of these approaches are predictor-corrector methods. This paper proposes a continuous block linear multistep method for direct solution of second order ordinary differential equations without resolving such equations to a system of first order ordinary differential equations. This approach circumvent the problem of developing an appropriate corrector formulae and it equally reduces computation error because the method is self-starting which need no additional information in getting the starting values which may alter the accuracy of the result. Interpolation and collocation were done at some selected grid points to derive the method.

Keywords: Collocation, Interpolation, Multistep, Convergence, predictor – corrector.

1. INTRODUCTION

The numerical solution of differential equations cannot be overemphasized. The direct solution of some differential equations are intractable hence there is need to circumvent this problem. By so doing, an appropriate (numerical) solution to the differential equation is soughed for. So many methods have been proposed in solving differential equations numerically.

Some of those methods include Runge – Kutta method, Euler's method, Heun's method to mention just a few. All these methods are single step methods which their accuracy is limited [11]. However, researchers have tried to improve upon this by proposing multistep method that is linear multistep method which is basically predictor-corrector in nature. Some of these methods includes Adams Moulton and Adams Bashforth method, Backward Differentiation Formula and Hybrid methods. These methods are specifically developed to solve first order ordinary differential equations.

In solving higher order ODEs, it is often the practice to reduce such an equation to a system of first order ODE and then solve. This method is too laborious to execute. In order to bypass this process, people have come up with so many methods that can be used in solving higher order ODEs directly without need to resolve such an equation to a system of first order ODE. Some of the authors that proposed this approach and have implemented it include [1, 2, 3, 6, 7, 8]. Basically all these methods are predictor-corrector in nature which often affect the accuracy of the result because an additional information is needed to get the starting value. In order to overcome the shortcoming of predictor-corrector method, researchers have proposed another method known as Block Linear Multistep method. This method does not need any additional starting value, in order words it is a self-starting method thereby increasing the accuracy of the method. Such authors that proposed this method and have applied it include [3, 4, 5, 10, 12]

In this presentation, a block linear multistep method is proposed using Chebyshev Polynomial as basis function. The choice of Chebyshev polynomial as basis function is as a result of its accuracy among all other monomials in the interval [-1, 1], it has the least maximum magnitude of $2^{1-n}\, T_n(x)$, it is also known to oscillate with equal amplitude through their range of definition [9].

2. THE METHODOLOGY

The general Linear Multistep Method for direct solution of Second Order Ordinary Differential Equations is of the form

$$\sum_{j=0}^{k} \alpha_j\, y_{n+j} = h^2 \sum_{j=0}^{k} \beta_j\, y_{n+j} \qquad (1)$$

In which α_j *and* β_j are not necessarily zero. In order to achieve our goal, we make use of Chebyshev Polynomial as basis function based on the reasons adduced above and this is defined as

$$y_{(x)} = \sum_{r=o}^{k} a_r\, T_r(x) \qquad (2)$$

Where
$T_r(x) = 2xT_n(x) - T_{n-1}(x)$ In which $T_0(x) = 1$ *and* $T_1(x) = x$ (3)

In order to derive our method which is a six step method, we let $k = 8$ in equation (2) . Evaluating equation (2) in the interval $x_n \le x \le x_{n+6}$ yields

$$y_{(x)} = \sum_{r=0}^{k} a_r\, T_r \left\{\frac{2x-2kh-6h}{6h}\right\} \qquad (4)$$

Letting $k = 8$ in equation (2) leads to the polynomial equation

$$y_{(x)} = a_0 + a_1 x + a_2(2x^2 - 1) + a_3(4x^3 - 3x) + a_4(8x^4 - 8x^2 + 1) + a_5(16x^5 - 20x^3 + 5x) + a_6(32x^6 - 48x^4 + 18x^2 - 1) + a_7(64x^7 - 112x^5 + 56x^3 - 7x) + a_8(128^8 - 256x^6 + 160x^4 - 32x^2 + 1) \quad \ldots\ldots (5)$$

Differentiating equation 5 twice yields

$$y''_{(x)} = 4a_2 + 24\, a_3 x + a_4(96x^2 - 16) + a_5(320x^3 - 120x) + a_6(960x^4 - 576x^2 + 36) + a_7(2688x^5 - 2240x^3 + 336x) + a_8(7168x^6 - 7680x^4 + 1920x^2 - 64) \quad \ldots\ldots (6)$$

Interpolating equation (5) at $x = x_n$ *and* x_{n+1} and collocating equation (6) at $x_{n+p}, p = 0(1)6$ and by making use of equation (4) leads to the matrix equation $Ax = B$

Where

$$A = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 6561 & -4371 & -729 & 5346 & -6399 & 3186 & 2151 & -6054 & 1344 \\ 0 & 0 & 4 & -24 & 80 & -200 & 420 & -784 & 5921 \\ 0 & 0 & 2916 & -1164 & 19440 & 10800 & -22140 & 62496 & -71744 \\ 0 & 0 & 2916 & -5832 & -3888 & 20520 & -1772 & -29232 & 46912 \\ 0 & 0 & 4 & 0 & -16 & 0 & 36 & 0 & -64 \\ 0 & 0 & 2916 & 5832 & -3888 & -20520 & -1772 & 29232 & 46912 \\ 0 & 0 & 2916 & 1164 & 19440 & 10800 & -22140 & -62496 & -71744 \\ 0 & 0 & 4 & 24 & 80 & 200 & 420 & 784 & 5921 \end{bmatrix}$$

$$x = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{pmatrix} \text{ and } B = \begin{pmatrix} y_n \\ 6561 y_n \\ f_n \\ 729 f_{n+1} \\ 729 f_{n+2} \\ f_{n+3} \\ 729 f_{n+4} \\ 729 f_{n+5} \\ f_{n+6} \end{pmatrix}$$

Solving these set of equations using Maple 9 software yields the following

$$a_0 = 3y_{n+1} - 2y_n + \frac{h^2}{46448640}\{747619f_n + 10623534f_{n+1} + 6445773f_{n+2} + 6298948f_{n+3} + 1822413f_{n+4} + 1161774\, f_{n+5} - 5021x_{n+6}\}$$

$$a_1 = 3y_{n+1} - 3y_n + \frac{h^2}{5806080}\{124283f_n + 1691652f_{n+1} + 1137447f_{n+2} + 1200320f_{n+3} + 428793f_{n+4} + 254268\, f_{n+5} + 1637x_{n+6}\}$$

$$a_2 = \frac{h^2}{30720}\{37f_n + 918f_{n+1} + 1755f_{n+2} + 2260f_{n+3} + 1755f_{n+4} + 918\, f_{n+5} + 37x_{n+6}\}$$

$$a_3 = \frac{h^2}{30720}\{-37f_n - 612f_{n+1} - 585f_{n+2} + 585f_{n+4} + 612\, f_{n+5} + 37\, x_{n+6}\}$$

$$a_4 = \frac{h^2}{61440}\{59f_n + 702f_{n+1} - 459f_{n+2} - 604f_{n+3} - 459f_{n+4} + 702\, f_{n+5} + 59f_{n+6}\}$$

$$a_5 = \frac{h^2}{10240}\{-9f_n - 36f_{n+1} + 99f_{n+2} - 99f_{n+4} + 36\, f_{n+5} + 9f_{n+6}\}$$

$$a_6 = \frac{h^2}{71680}\{39f_n + 18f_{n+1} - 423f_{n+2} + 732f_{n+3} - 423f_{n+4} + 18\, f_{n+5} + 39f_{n+6}\}$$

$$a_7 = \frac{h^2}{71680}\{-27f_n + 108f_{n+1} - 135f_{n+2} + 135f_{n+4} - 108\, f_{n+5}\, 27f_{n+6}\}$$

$$a_8 = \frac{h^2}{573440}\{81f_n - 486f_{n+1} + 1215f_{n+2} - 1620f_{n+3} + 1215f_{n+4} - 486\, f_{n+5} + 81f_{n+6}\}$$

Substituting these values of a's into equation (5) leads to the continuous multistep method.

$$y(x) \cong 3(x+1)y_{n+1} - (3x+2)y_n + \tfrac{h^2 f_n}{46448640}(717568 + 1080448x - 387072x^3 + 193536x^4 + 1306368x^5 - 870912x^6 - 1119744x^7 + 839808x^8) + \tfrac{h^2 f_{n+1}}{46448640}(9715200 + 15002880x + 3483648x^3 - 2612736x^4 - 10450944x^5 + 10450944x^6 + 4478976x^7 - 5038848x^8) + \tfrac{h^2 f_{n+2}}{46448640}(3817728 + 14610816x - 17418240x^3 + 26127360x^4 + 16982784x^5 - 33965568x^6 - 5598720x^7 + 12597120x^8) + \tfrac{h^2 f_{n+3}}{46448640}(1819648 + 9602560x + 23224320x^2 - 47416320x^4 + 48771072x^6 - 16796160x^8) + \tfrac{h^2 f_{n+4}}{46448640}(-805632 - 2080896x + 17418240x^3 + 26127360x^4 - 1692784x^5 - 33965568x^6 + 5598720x^7 + 12597120x^8) + \tfrac{h^2 f_{n+5}}{46448640}(253440 + 564480x - 3483648x^3 - 2612736x^4 + 10450944x^5 + 10450944x^6 - 4478976x^7 - 5038848x^8) + \tfrac{h^2 f_{n+6}}{46448640}(-35072 - 73088x + 387072x^3 + 193536x^4 - 1306368x^5 - 870912x^6 + 1119744x^7 + 839808x^8) \qquad (7)$$

Evaluating equation (7) at $x_{n+p}; p = 1(1)6$ and making use of equation (4) leads to the Block method

$$y_{(x)} = AY_m - h^2 BF_m \tag{8}$$

Where

$$A = \begin{pmatrix} 6 & -5 \\ 5 & -4 \\ 4 & -3 \\ 3 & -2 \\ 2 & -1 \end{pmatrix}, \qquad Y_m = \begin{pmatrix} y_{n+1} \\ y_n \end{pmatrix}$$

$$B = \begin{bmatrix} \frac{1}{36228}(1375 \quad 19554 \quad 13401 \quad 15004 \quad 6177 \quad 4770 \quad 199) \\ \frac{1}{54432}(1669 \quad 23250 \quad 15207 \quad 15004 \quad 4371 \quad 1074 \quad -95) \\ \frac{1}{90720}(2089 \quad 28878 \quad 16383 \quad 13828 \quad -1257 \quad 654 \quad -360) \\ \frac{1}{181440}(2803 \quad 37950 \quad 14913 \quad 7108 \quad -3147 \quad 990 \quad -137) \\ \frac{1}{544320}(4315 \quad 53994 \quad -2307 \quad 7948 \quad 4827 \quad 1578 \quad -221) \end{bmatrix}$$

$$F_m = \begin{pmatrix} f_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \\ f_{n+6} \end{pmatrix}$$

3. ANALYSIS OF THE METHOD

Here the order and error constant of the method shall be analysed. At this juncture it is very important to know the condition by which a linear multistep method will converge. For a linear multistep method to converge, it must be consistent and zero stable, in which consistency controls the magnitude of the local truncation error committed at each stage of the calculation while zero stability controls the manner in which the error is propagated as the calculation proceeds.

DEFINITION: A linear multistep method is said to be zero stable if no root of the first characteristics polynomial $\rho(\tau)$ has modules greater than one, and if every roots with modulus one is simple.

DEFINITION: Let the linear operator $\mathcal{L}\{y(x); h\}$ associated with the Linear Multistep method

$$\mathcal{L}\{y(x); h\} = \sum_{j=0}^{k} \left[\alpha_j \, y(x + jh) - h^2 \beta_j y''(x + jh)\right] \tag{9}$$

Where $y(x)$ is an arbitrary function, continuously differentiable function in the interval $[a, b]$.

The Tailor's series expansion of equation (8) about the point x yields

$$h[y(x)\,;h] = C_0 y(x) + C_1 y(x) + C_2 y(x) + C_3 y(x) + \ .\ .\ .\ + C_q h^q y^q(x)$$

where $C_0 = \sum_{j=0}^{k} \alpha_j$, $\quad C_1 = \sum_{j=1}^{k} j\alpha_j$, $\quad , C_2 = \frac{1}{2!}\sum_{j=1}^{k} j^2\alpha_j - \sum_{j=0}^{k} \beta_j$

$$C_{q=} \frac{1}{q!}\sum_{j=1}^{k} j^q \alpha_j - \frac{1}{(q-2)!}\sum_{j=1}^{k} j^{q-2}\beta_j$$

The Linear Multistep method is of order p if $C_0 = C_1 = C_2 = \ .\ .\ .\ = C_{p+1} = 0\ and\ C_{p+2} \neq 0$. therefore C_{p+2} is the error constant. On investigating the order of our derived method, that is equation (8) it was found that it is of order 8 with error constant of $(1.046x10^{-4}, 1.26x10^{-3}, 1.301x10^{-3}\ , 2.067x10^{-4}\ , 5.12310^{-4})^T$ in like manner it was discovered that the methods derived are both consistent and zero stable having fulfill the conditions stated above. Therefore this guarantees that the methods will surely converge.

4. NUMERICAL EXPERIMENT:

At this point, the interest lies in the numerical computation in which the method derived shall be applied in solving some second order ordinary differential equations directly and the computed result shall be compared with the analytical solution to see the level of accuracy of the derived method. Also the result gotten from this method shall be compared with that of ( ) In this wise the following equations were considered.

ILLUSTRATION I: Determine the solution to the differential equation
$x(x-1)y'' - xy' + y = 0,\ y(1) = 3$ in the interval $0 < x \le 1\ with\ h = 0.1$. Given that $y_1(x) = x$ is one of the solutions to the differential equation.

The analytical solution is

$$y(x) = x + xln\,x + 1$$

The computed result in as shown in Table I below

ILLUSTRATION II: Compute the numerical solution to the differential equation
$x^2y'' - x(x-2)y' + (x+2)y = 0, y(1) = 2$ in the interval $0 \le x \le 1\ with\ h = 0.1$. Given that $y_1(x) = x$ is one of the solutions to the differential equation. The analytical solution is
$y(x) = 1.28x + xe^x$. The computed result in as shown in Table II below

ILLUSTRATION III: Solve the differential equation
$y'' - x(y')^2 = 0, y(0) = 1, y'(0) = 0.5$ in the interval $0 \le x \le 1\ with\ h = 0.1$. The theoretical solution is $y(x) = 1 + \ln\left(\frac{2+x}{2-x}\right)^{0.5}$. The solution is as shown in Table III below

ILLUSTRATION IV: Find the solution to the differential equation $x^2y'' - 4xy' + 6y = 0, y(1) = 2, y'(1) = 5$ in the interval $0 \leq x \leq 1$ $with$ $h = 0.1$. The theoretical solution is $y(x) = x^2(1 + x)$. The computed result is as shown in Table IV below.

Table I. Numerical Computation To Illustration I

X	Exact Solution	Computed Result	Absolute Error I	Absolute Error Ii
0.1	0.86974149	0.8697413205	1.695 E - 07	2.5634E - 06
0.2	0.878112417	0.8781123481	6.89 E- 08	7.872E - 07
0.3	0.938808158	0.9388082462	8.82 E -08	8.1894E - 07
0.4	1.033483707	1.0334857063	1.999 E - 06	2.6743E – 06
0.5	1.153426410	1.1534269221	5.12 E - 07	5.2307E – 06
0.6	1.293504626	1.2935044381	1.88 E - 07	1.9912E - 07
0.7	1.450327539	1.4503277436	2.04 E - 07	2.5607E - 07
0.8	1.621485159	1.6214853781	2.19 E - 07	8.6741E – 07
0.9	1.805175536	1.8051753941	1.42 E - 07	6.2298E – 07
1.0	2.0000000000	1.9996738176	3.26183 E - 04	7.2453E - 04

ABSOLUTE ERROR I IS ABSOLUTE DIFFERENCE IN EXACT VALUE AND COMPUTED RESULT USING THE DERIVED METHOD

ABSOLUTE ERROR II IS ABSOLUTE DIFFERENCE IN EXACT VALUE AND USING THE DERIVED METHOD IN [8]

Table II. Numerical Computation To Illustration II

X	EXACT SOLUTION	COMPUTED RESULT	ABSOLUTE ERROR I	ABSOLUTE ERROR II
0	0.000000000000	0.0000006794	6.794 E - 07	8.5214E - 06
0.1	0.238517091	0.2385173642	2.732 E - 07	5.2271E - 07
0.2	0.500280551	0.5002809247	3.737 E - 07	4.8201E - 07
0.3	0.788957642	0.7889592731	1.6311 E - 06	2.8821E – 06
0.4	1.108729878	1.1087298142	6.400 E - 08	6.9237E - 07
0.5	1.464360635	1.4643608198	1.84 E - 07	1.992E - 06
0.6	1.861271280	1.8612791258	7.845 E - 06	9.5217E - 06
0.7	2.305626895	2.3056265171	3.78 E - 07	7.5617E - 07
0.8	2.804432743	2.8044309891	1.754 E - 06	6.3358E - 06
0.9	3.3656428	3.3656467415	3.941 E - 06	8.5625E - 06
1.0	3.998281828	3.9982867864	4.958 E - 06	8.0126E - 06

ABSOLUTE ERROR I IS ABSOLUTE DIFFERENCE IN EXACT VALUE AND COMPUTED RESULT USING THE DERIVED METHOD

ABSOLUTE ERROR II IS ABSOLUTE DIFFERENCE IN EXACT VALUE AND USING THE DERIVED METHOD IN [8]

TABLE III. Numerical Computation To Illustration III

X	EXACT SOLUTION	COMPUTED RESULT	ABSOLUTE ERROR I	ABSOLUTE ERROR II
0	1.00000000000	0.9998978214	1.021786E- 04	3.8721E - 04
0.1	1.316359698	1.3163598927	1.940 E -07	6.2972E - 07
0.2	1.447962828	1.4479629854	1.570 E -07	5.8276E - 07
0.3	1.549800757	1.5498009874	2.30 E - 07	2.8821E - 06
0.4	1.636761422	1.6367619846	5.62 E - 07	7.3201E - 07
0.5	1.714720661	1.7147206086	5.30 E - 08	3.4572E - 07
0.6	1.786790448	1.7867905091	6.10 E - 08	8.9247E - 07
0.7	1.854919592	1.8549195032	8.90 E -08	7.5617E - 07
0.8	1.920487838	1.9204878812	4.30 E - 08	3.3455E - 07
0.9	1.969400557	1.9694005247	3.30 E - 08	4.0915E - 07
1.0	2.048147074	2.048147952	8.78032 E - 07	8.0126E - 06

ABSOLUTE ERROR I IS ABSOLUTE DIFFERENCE IN EXACT VALUE AND COMPUTED RESULT USING THE DERIVED METHOD

ABSOLUTE ERROR II IS ABSOLUTE DIFFERENCE IN EXACT VALUE AND USING THE DERIVED METHOD IN [8]

TABLE IV. Numerical Computation To Illustration IV

X	EXACT SOLUTION	COMPUTED RESULT	ABSOLUTE ERROR I	ABSOLUTE ERROR II
0	0.0000000000	0.0007892614	7.892614 E - 04	9.2721E - 04
0.1	0.011	0.0089624156	2.0375844 E – 03	8.1072E - 03
0.2	0.048	0.0496861321	1.6861321 E – 03	5.2876E - 03
0.3	0.117	0.1175932413	5.932413 E – 04	8.7221E - 04
0.4	0.224	0.2240621315	6.21315 E – 05	1.3201E - 04
0.5	0.375	0.3755132492	5.132492 E – 04	7.6272E - 04
0.6	0.576	0.5760021314	2.1314 E – 06	8.0137E - 06
0.7	0.833	0.8330005493	5.493 E – 07	8.0217E - 07
0.8	1.152	1.1520007813	7.81 E – 07	9.1635E - 07
0.9	1.539	1.5390010214	1.021 E – 06	5.7695E - 06
1.0	2.0	1.9990802436	9.19757 E - 04	1.9626E - 03

ABSOLUTE ERROR I IS ABSOLUTE DIFFERENCE IN EXACT VALUE AND COMPUTED RESULT USING THE DERIVED METHOD

ABSOLUTE ERROR II IS ABSOLUTE DIFFERENCE IN EXACT VALUE AND USING THE DERIVED METHOD IN [8]

5. DISCUSSION OF RESULT

At this point, the discussion of results based on the solution to the differential equations considered shall be discussed. First and foremost, from Table I, it was observed that the analytical solution and the computed solution to Illustration I are very close. This shows that the method used (that is the derived method) yields a much closed result with the analytical solution of the deferential equation considered. So also, the results shows a better approximation when compared with that of method [8]. In like manner, Table II shows the comparison between the analytical (theoretical solutions) and the numerical (approximate solution) of the differential equation considered. It was evident that the result generated using the derived method is in tandem with the true solution of the differential equation and is equally better than that of method [8]. Illustrations III and IV equally show that the derived method is highly accurate in solving second order ordinary differential equations based on the output of the result and gives a better approximate value that that of method [8]

At this juncture, it has been shown clearly that Second Order Ordinary Differential Equations can be solved numerically without necessarily resolving such an equation to system of First Order Ordinary Differential Equations. Carefully selecting the points of collocation of differential equations and interpolating the polynomial equation will surely lead to desired Linear Multistep method in solving differential equations. The proposed block method is desirable in which it does not need to get a corrector formula before it can be implemented unlike the popular predictor – corrector method which is often very laborious to implement. Another desirability of the Block method is that it is self-starting which requires no additional value to start the computation thereby reducing accumulative error as the computation progresses as it was shown in the Tables I to IV when compared the results with that of method [8] which is a predictor- corrector method.

In conclusion, we hereby recommend that further research should be carried out on deriving another Block method which is going to have a smaller error constant and higher accuracy, also to consider other polynomials such as Hermite polynomial, Legendre Polynomial, Canonical polynomial among others as basis function in deriving Multistep methods.

REFERENCES

(1) Adee, S. O., Onumanyi, P., Serisena, U. V., Yahaya, Y. A., (2005) " A Note on Starting Numerov's Method More Accurately by an Hybrid Formula of Order Four for an Initial Value Problem" . *Journal of Computational and Applied Mathematics, Vol. 175, No. 4.*

(2) Adesanya, A. O. , Anake, T. A., and Udoh, M. O., (2008)" Improved Continuous Method for Direct Solution of General Second Order ordinary Differential Equations". *Journal of Nigeria Association of Mathematical Physics, Vol.13*

(3) Alabi, M. O. (2008) "A Continuous Formulation of Initial Value Solvers with Chebyshev Polynomial as Basis Functions in a Multistep Collocation Technique" (Ph. D Thesis).

(4) Alabi, M. O.,Adewoye, K. S., and Olaleye, O.A. (2017) "Derivation of Block Continuous Initial Value Integrators Using Legendre Polynomial as Perturbation Term via Power Series Method" *Journal of New Trends in Sciences and Technology, Vol 2 No.1.*

(5) Alabi, M.O., Adewoye, K.S. and Babatunde, O.Z. (2018) "Integrator Block Off – Grid Points Collocation Method for Direct Solution of Second Order Differential Equations Using Chebyshev Polynomials as Basis Function" *International Journal of Mathematics and Statistics Studies, Vol 7, No. 1, pp13 – 21*

(6) Awoyemi, D.O. (2006). " A new Six Order Algorithm for the General Solution Of Second Order Ordinary Differential Equation" . *International Journal of Computer Mathematics. Vol.77 No. 1 pp. 117 – 124*

(7) Awoyemi D. O. and Kayode, S. J. (2005) " A Maximal Order Collocation Method for Direct Solution on Initial Value Problems of General Second Order Ordinary Differential Equations." *Proceedings of the Conference Organized by the National Mathematical Centre, Abuja.*

(8) Awoyemi D. O. and Kayode, S. J. (2003). " An Optimal Order Collocation Method for Direct Solution of Initial Value Problems of General Second Order Ordinary Differential Equations". FUTAJEET, Vol.3, pp. 33 - 40

(9) Awoyemi, D. O. and Idowu, M. O. (2005) " A Class of Hybrid Collocation Method for Third Order Ordinary Differential Equation". *International Journal of Computer Mathematics Vol.82, No. 10*

(10) Fox, L. and Parker, I. B. (1968) "Chebyshev Polynomials in Numerical Analysis". Oxford University Press, London.

(11) Jator, N. S. (2007) " A Six Order Linear Multistep Method for Direct Solution of $y'' = f(x, y, y')$". *International Journal of Pure and Applied Mathematics. Vol. 40, No1.*

(12) Lambert, J.D. (1973) "Computational Methods in Ordinary Differential Equations". John Wiley and Sons, London.

(13) Yahaya, Y. A. and Badmus, A. M. (2009) " A class of Collocation Methods for General Second Order Differential Equation". *African Journal of Math and Computer Research.*