Derivation of a Numerical Scheme to find any Root of any Real Number k using Newton Raphson Iterative Method

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ABSTRACT

This research paper succinctly explains the derivation of a numerical scheme from the Newton-Raphson's iterative method. A juxtapose of this newly developed scheme with Newton-Raphson's shows an improvement of finding better approximations to any root of a real number.

Keywords: Newton Raphson method, linear function, Iteration.

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1. INTRODUCTION

Numerical analysis is the study of algorithms that use numerical approximation (as opposed to general symbolic manipulations) for the problems of mathematical analysis (as distinguished from discrete mathematics). Numerical analysis naturally finds applications in all fields of engineering and the physical sciences, but in the 21st century also the life sciences and even the arts have adopted elements of scientific computations. Ordinary differential equations appear in celestial mechanics (planets, stars and galaxies); numerical linear algebra is important for data analysis; stochastic differential equations and Markov chains are essential in simulating living cells for medicine and biology. Before the advent of modern computers numerical methods often depended on hand interpolation in large printed tables. Since the mid-20th century, computers calculate the required functions instead. These same interpolation formulas nevertheless continue to be used as part of the software algorithms for solving differential equations.

One of the earliest mathematical writings is a Babylonian tablet from the Yale Babylonian Collection (YBC 7289), which gives a sexagesimal numerical approximation of the square root of 2, the length of the diagonal in a unit square. Being able to compute the sides of a triangle (and hence, being able to compute square roots) is extremely important, for instance, in astronomy, carpentry and construction. Numerical analysis continues this long tradition of practical mathematical calculations. Much like the Babylonian approximation of the square root of 2, modern numerical analysis does not seek exact answers, because exact answers are often impossible to obtain in practice. Instead, much of numerical analysis is concerned with obtaining approximate solutions while maintaining reasonable bounds on errors.
One of the most useful schemes in Numerical Analysis is the Newton-Raphson's iterative method. It was developed by Isaac Newton (1643 – 1727) and Joseph Raphson (1648 – 1715). The Newton-Raphson method, is a powerful technique for solving equations numerically. Like so much of the differential calculus, it is based on the simple idea of linear approximation. The Newton Method, properly used, usually homes in on a root with devastating efficiency.

Many researchers have developed numerical schemes for solving initial value problems. Such as: Ayinde S.O et.al (2015), Butcher, J. C. (2003), Fatunla, S.O. (1988), Dahlquist, G. and Bjorck, A.(1974) in which standard finite difference schemes were developed. Obayomi, A.A.(2012) also worked on some approximation techniques which was used to derive qualitatively stable non-standard finite difference schemes.

Fatunla, S. O. (1976), Fatunla, S.O. (1988), Ibijola, E. A. (1997), Lambert, J. D. (1973), Ibijola E.A and R.B Ogunrinde (2010**)** have developed numerical schemes to generate the numerical solutions of initial value problem (IVP) of the form. R. B. Ogunrinde and K.I Oshinubi.(2017) used Adomian Decomposition scheme to solve some Differential equation and model.

2. AN OVERVIEW OF THE NUMERICAL SCHEME

In Newton-Raphson's iterative method, we must first assume that the function is differentiable, i.e. that function $f f$ has a definite slope at each point. Hence at any point $(x_0, f(x_0))$, we can calculate the tangent $f'(x_0)$, which is a fairly good approximation to the curve at that point. Alternatively, we can see that linear function

$$l(x) = f'(x_0)(x - x_0) + f(x_0)$$

is quite close to function f near point x_0, and at x_0, the functions l

and f give the same value.

Hence, we take it that the solution to the problem $l(x) = 0$

will give a fairly good approximation to the problem $f(x) = 0$.

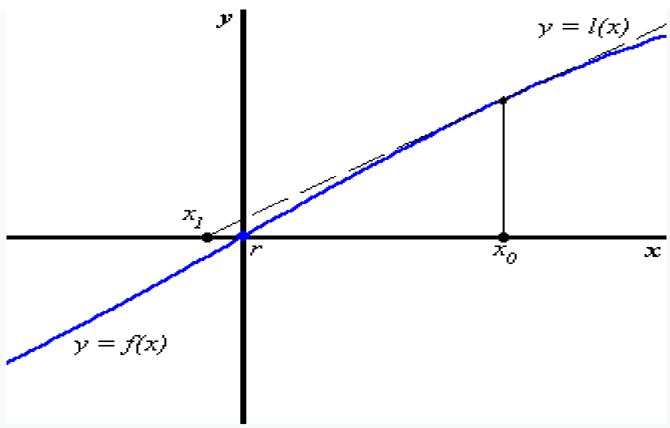

Fig. 1: An illustration showing Newton Raphson's iterative method.

The zero
of $l(x) l(x)$

can be easily found:

$$l(x) = 0$$

$$f'(x_0)(x - x_0) + f(x_0) = 0$$

$$x = x_0 - \left(\frac{f(x_0)}{f'(x_0)}\right)$$

This can be done repeatedly to produce points with the following equation:

$$x_{n+1} = x_n - \left(\frac{f(x_n)}{f'(x_n)}\right)$$

From x_1, we produce a new estimate x_2. From x_3, we produce a new estimate x_3. We go on until we are `close enough' to r or until it becomes clear that the same solution comes repeatedly. The above general style of proceeding is called *iteration.* Of the many iterative root-finding procedures, the Newton-Raphson method, with its combination of simplicity and power, is the most widely used.

3. AN OVERVIEW OF THE DERIVED SCHEME

After a thorough understanding of how to find the square root of a real number k using the Newton Raphson iterative method, I observed we could use the same method to find cube root, fourth root, fifth root, ... , p^{th} root of a real number k. Firstly, I derived the formula to find the square root of a real number k again.

Let $x = \sqrt{k} \Longrightarrow x^2 = k \Longrightarrow x^2 - k = 0$

If $f(x) = x^2 - k$, $f(x_0) = {x_0}^2 - k$

Then $f(x_n) = {x_n}^2 - k \Longrightarrow f'(x_n) = 2x_n$

Using Newton-Raphson iterative method - $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$$x_{n+1} = {x_n}^2 - \frac{{x_n}^2 - k}{2x_n}$$

$$= \frac{1}{2}x_n + \frac{k}{2x_n}$$

$$= \frac{1}{2}(x_n + \frac{k}{x_n})$$

Then for a cube root of k

Let $x = \sqrt[3]{k} \Longrightarrow x^3 = k \Longrightarrow x^3 - k = 0$

If $f(x) = x^3 - k$, $f(x_0) = {x_0}^3 - k$

Then $f(x_n) = {x_n}^3 - k \Longrightarrow f'(x_n) = 3{x_n}^2$

Using Newton-Raphson iterative method - $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$$x_{n+1} = {x_n}^3 - \frac{{x_n}^3 - k}{3{x_n}^2}$$

$$= \frac{1}{3}(2x_n + \frac{k}{{x_n}^2})$$

Applying the same method, I derived the schemes for cube root and fourth root of a real number k.

The results are:

$\frac{1}{4}(3x_n + \frac{k}{{x_n}^3})$ and $\frac{1}{5}(4x_n + \frac{k}{{x_n}^4})$ for fourth and fifth roots respectively for any real number k.

Collecting the results together, I derived a general expression to find the nth root of any real number k as follows:

If $x = \sqrt[p]{k}$ where $k \in \mathbb{R}$ and $p \in \mathbb{R}$, then the derived scheme is

$$x_{n+1} = \frac{1}{p}\left\{(p-1)x_n + \frac{k}{{x_n}^{p-1}}\right\}$$

To proof the validity of the formulae, I used it to find $\sqrt[3]{9}$, $\sqrt[10]{100}$ and $\sqrt[5]{64}$ using $x_0 = 1.5, 1.2$ and 0.8 respectively. The results gotten were $2.08, 1.58$ and 2.29 respectively and were proved to be accurate using a calculator.

4. APPLICATION

$$X_n = \sqrt[P]{k}$$

$$X_{n+1} = 1/P\{(P-1)X_n + \frac{K}{X_n^{(P-1)}}\}$$

Where P and K are constants

X_o = Intial value of X_n

ENTER VALUES FOR P, K & Xn IN C10, D10 & E10 RESPECTIVELY. ALL OTHER BOXES SHOULDN'T BE TOUCHED

Iteration	P	K	Xn	(1/P)*((P-1)*x(n-1)+(K/(x(n-1)^(P-1)))
1	3	64	0.09	2633.804856
2	3	64	2633.804856	1755.869907
3	3	64	1755.869907	1170.579945
4	3	64	1170.579945	780.3866455
5	3	64	780.3866455	520.2577987
6	3	64	520.2577987	346.8386113
7	3	64	346.8386113	231.2259182
8	3	64	231.2259182	154.1510111
9	3	64	154.1510111	102.7682385
10	3	64	102.7682385	68.51417898
11	3	64	68.51417898	45.68066394
12	3	64	45.68066394	30.46399933
13	3	64	30.46399933	20.33232002
14	3	64	20.33232002	13.60648419
15	3	64	13.60648419	9.186219809
16	3	64	9.186219809	6.376951186
17	3	64	6.376951186	4.775905922
18	3	64	4.775905922	4.119229223
19	3	64	4.119229223	4.00341774
20	3	64	4.00341774	4.000002917
21	3	64	4.000002917	4
22	3	64	4	4

Iteration	P	K	Xn	(1/P)*((P-1)*x(n-1)+(K/(x(n-1)^(P-1)))
23	3	64	4	4
24	3	64	4	4
25	3	64	4	4
26	3	64	4	4
27	3	64	4	4
28	3	64	4	4
29	3	64	4	4
30	3	64	4	4
31	3	64	4	4
32	3	64	4	4
33	3	64	4	4
34	3	64	4	4
35	3	64	4	4
36	3	64	4	4
37	3	64	4	4
38	3	64	4	4
39	3	64	4	4
40	3	64	4	4
41	3	64	4	4
42	3	64	4	4
43	3	64	4	4
44	3	64	4	4
45	3	64	4	4
46	3	64	4	4
47	3	64	4	4
48	3	64	4	4
49	3	64	4	4
50	3	64	4	4

5. CONCLUSION

The new scheme can be applied extensively much better than the Newton-Raphson iterative method because of its uniqueness in solving any root of any real number k.

It was also observed that the new scheme converges easily.

A revisit to Kepler's equation for solutions of planetary orbit, using this derived scheme, could spark new solutions or ideas to continue improvement on its application.

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