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Comparative Study of the Efficiency of Some Shrinkage Estimators on the Seemingly Unrelated Regression Model

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ABSTRACT

This study explored the efficiency of six different shrinkage estimators: Firinguetti (R_{SF}), Alkhamisi and Shukur-Median (R_{SK}), Alkhamisi and Shukur-Maximum (R_{AS}), Hoerl and Kennard-Maximum (R_{MSHK}), Hoerl *et al*-Harmonic mean (R_{MSham}), and Kibria $(R_{Msarith})$ on the Seemingly Unrelated Regression (SUR) model. A three-equation joint model was considered with different correlation levels among the explanatory variables ($\boldsymbol{\rho}_{x_ix_j}$) and contemporaneous correlation levels (ρ_{ε_M}) among the equations. Samples sizes 20, 30, 50 and 100 replicated 10000 times in turn were considered for the simulation study. Results from the study revealed that the Trace Mean Square Error (TMSE) values for all the estimators decreased as the sample sizes increased when the different correlation levels among the explanatory variables were considered. When n = 20, ρ_{ε_M}= 0.9, the estimator R_{AS} had the best performance in terms of the TMSE criterion compared to the remaining estimators for all the cases of $\boldsymbol{\rho}_{x_ix_j}$. When n = 30, ρ_{ε_M}= 0.9, the estimator R_{SK} had the best performance in terms of the TMSE criterion compared to the remaining estimators for $\boldsymbol{\rho}_{x_ix_j}$ = 0.0, 0.9, while the estimator R_{AS} gave the best performance in terms of the TMSE criterion compared to the remaining estimators for $\boldsymbol{\rho}_{x_ix_j}$ = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8. When n = 50, ρ_{ε_M}= 0.9, the estimator R_{SK} had the best performance in terms of the TMSE criterion compared to the remaining estimators for $\boldsymbol{\rho}_{x_ix_j}$ = 0.0, 0.1, 0.2, 0.3, 0.4, while the estimator R_{AS} had the best performance in terms of the TMSE criterion compared to the remaining estimators for $\boldsymbol{\rho}_{x_ix_j}$ = 0.5, 0.6, 0.7, 0.8, 0.9. When n = 100, ρ_{ε_M}= 0.8, the estimator R_{SK} outperformed the other estimators in all cases except for $\boldsymbol{\rho}_{x_ix_j}$ = 0.7, 0.8, 0.9. Conclusively, Alkhamisi and Shukur-Median (R_{SK}) outperformed other estimators, followed by Alkhamisi and Shukur-Maximum (R_{AS}).

Keywords: Multicollinearity, Ridge regression, Seemingly unrelated regression, Shrinkage estimators, Trace mean square error.

1. INTRODUCTION

The Seemingly Unrelated Regression (SUR) estimator which deals with a system of multivariate regression equations when error variables are contemporaneously correlated has recently received great attention from researchers, especially on cross-sectional studies (Zellner, 1962). It has been widely applied for the efficient joint estimation of regression parameters rather than the equation-by-equation estimation method of the Ordinary Least Squares (OLS). Zellner and Theil (1962), Zellner (1971), Kunitomo (1977), Adebayo (2003), Alaba and Akinrelere (2019), Alaba and Lawal (2019), Alaba *et al* (2019), Alaba *et al* (2010) have found that the estimation procedure of SUR is more efficient than the OLS; the gain in efficiency would be appreciated if the contemporaneous correlation between each pair of the disturbances in the SUR system of equations is $\left|\rho_{e_i,e_j}\right|$ > 0.3 and explanatory variables in different equations are uncorrelated.

Firinguetti (1997) pointed out that multicollinearity in a system of SUR is far more complex than in a classical linear regression model. The damaging effects of multicollinearity on the GLS estimator and because under some circumstances the estimator will not generate any efficiency benefits over OLS, biased estimators should be taken into account as a way to lower the variance of parameter estimates. The Ridge regression estimator offers one of such possibilities.

Ridge regression is used to examine multicollinear multiple regression data. Although it is a fundamental regularization method, its complicated scientific foundation explains why it is rarely generally applied. Hoerl first proposed in 1962 (Hoerl, 1962; Hoerl and Kennard, 1968) an estimator to curb inflation and overall unpredictability linked to the least squares estimator. Ridge regression prevents overfitting in regression analysis. Although least squares estimates are unbiased in the presence of multicollinearity, their enormous variations make them potentially inaccurate. Ridge regression lowers the standard errors adding a degree of bias to the regression estimates. A shrinkage tool used in ridge regression is called a ridge estimator.

A shrinkage estimator is a parameter that generates new estimators that have been reduced in size to provide a result that is more in line with the actual population parameters. When the data are multicollinear, a ridge estimator can shrink the least squares estimate for better estimate. There are different forms of shrinkage estimators which are connected to one another and have a similar form. This study explored the efficiency of six different shrinkage estimators: Firinguetti (R_{SF}), Alkhamisi and Shukur-Median (R_{SK}), Alkhamisi and Shukur-Maximum (R_{AS}), Hoerl and Kennard-Maximum (R_{MSHK}), Hoerl *et al*-Harmonic mean (R_{MSham}) and Kibria $(R_{Msarith})$ on the SUR model.

2. METHODOLOGY

The Seemingly unrelated regression (SUR) model with *M* equations and *T* observations is given as,

$$\underset{mn\times 1}{\begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}} = \underset{mn\times \sum k_i}{\begin{bmatrix} X_1 & \cdots & 0 \\ \vdots & & \\ 0 & \cdots & X_m \end{bmatrix}} \underset{\sum k_i \times 1}{\begin{bmatrix} \beta_1 \\ \vdots \\ \beta_m \end{bmatrix}} + \underset{mn\times 1}{\begin{bmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_m \end{bmatrix}} \tag{1}$$

where

y_i is an $mn \times 1$ vector of observations on the i^{th} response variable, X_i is a fixed $mn \times \sum k_i$ matrix of explanatory variables, β_i is a $\sum k_i \times 1$ vector of unknown regression parameters, ε_i is an $mn \times 1$ vector of disturbances such that $cov(\varepsilon) = \mathrm{E}\left[\varepsilon^{'}\varepsilon\right] \otimes I_n$, $E(\varepsilon) = 0$.

The OLS estimator of β in (1) is

$$\hat{\beta} = (X'X)^{-1} X^{'}Y \quad (2)$$

with

$$V(\hat{\beta}) = \left[X^{'}\left(\Sigma \otimes I_T\right)X\right]^{-1} \quad (3)$$

The Generalized Least Squares (GLS) estimator of $\hat{\beta}$ in (1) is;

$$\hat{\beta}_G = \left(X^{+}{}'\left(\Sigma^{-1} \otimes I_T\right)X^{+}\right)^{-1} X^{+'}\left(\Sigma^{-1} \otimes I_T\right)Y^{+} \quad (4)$$

$$V(\hat{\beta}_G) = \left[X^{+'}\left(\Sigma \otimes I_T\right)X^{+}\right]^{-1} \quad (5)$$

Hoerl and Kennard (1970) defined the general ridge estimator of β in (1) as

$$\hat{\beta}_{Ridge} = \left[X^{+}{}'\left(\Sigma^{-1} \otimes I\right)X^{+} + \psi R \psi'\right] X^{+'}\left(\Sigma^{-1} \otimes I\right)Y^{+} \quad (6)$$

where

$$\psi\psi' = \mathrm{I}, \quad R = \begin{pmatrix} R_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & R_M \end{pmatrix}, \quad \hat{R}_i = diag(1/\hat{\delta}_{iki}^{\;2}), \quad \hat{\delta}_i = \psi'\hat{\beta}_G$$

where R is a k × k matrix of non-negative elements. The ridge estimator in (6), however, excludes the information included in the correlation matrix of cross equation errors. The following transformation is more helpful to retain the information

$Y^* = \left(\Sigma^{-1/2} \otimes I_T\right)Y$, $X^* = \left(\Sigma^{-1/2} \otimes I_T\right)X$, and $e^* = \left(\Sigma^{-1/2} \otimes I_T\right)e$

Using this transformation, the model (1) becomes

$$Y^* = X^*\beta + e^* \quad (7)$$

The OLS estimator of $\hat{\beta}$ in (2), which is the GLS estimator of β in (4), and its ridge estimator in (6) are, respectively, as follows:

$$\hat{\beta} = (X^{*}{}'X^{*})^{-1} X^{*'}Y^{*} = \left(X'\left(\Sigma^{-1} \otimes I_T\right)X\right)^{-1} X^{'}\left(\Sigma^{-1} \otimes I_T\right)Y \quad (8)$$

$$\hat{\beta}_G = (X^{+*}{}'X^{+*})^{-1} X^{+*'}Y^{+*} = \left(X^{+}{}'\left(\Sigma^{-1} \otimes I_T\right)X^{+}\right)^{-1} X^{+'}\left(\Sigma^{-1} \otimes I_T\right)Y^{+} \quad (9)$$

$$\hat{\beta}_R = (X^{+*}{}' X^{+*} + \psi^{+} R \psi^{+}{}')^{-1} X^{+*'} Y^{+*} = \left(X^{+}{}' \left(\Sigma^{-1} \otimes I_T \right) X^{+} + \psi^{+} R \psi^{+}{}' \right)^{-1} X^{+'} \left(\Sigma^{-1} \otimes I_T \right) Y^{+} \quad (10)$$

Let Λ be a diagonal matrix of eigenvalues and ψ a matrix whose columns are eigenvectors of $X^{*}{}' X^{*}$.

The canonical version of the model in (7) is

$$Y^{*} = Z\alpha + e^{*} \quad (11)$$

where

The OLS estimator of α in (11) is

$$\hat{\alpha} = \left(Z'Z \right)^{-1} Z'Y^{*} \quad (12)$$

with its associated GLS and SUR Ridge regression parameter as follows

$$\hat{\alpha}_G{}^{+} = \left(Z^{+}{}' Z^{+} \right)^{-1} Z^{+}{}' Y^{+*}$$

$$\hat{\alpha}^{+}_{SUR}(R) = \left(Z^{+}{}' Z^{+} + \psi^{+} R \psi^{+}{}' \right)^{-1} Z^{+}{}' Y^{+*} \quad (13)$$

$$\text{MSE}(\hat{\alpha}_{SUR}) = \left(\Lambda + \text{R} \right)^{-1} (\Lambda + \text{R} \hat{\alpha}^{+}_{SUR}(R) \hat{\alpha}^{+}_{SUR}(\text{R})')(\Lambda + \text{R})^{-1} \quad (14)$$

$$\text{TMSE}(\hat{\alpha}^{+}_{SUR}) = \sum_{i=1}^{M} \sum_{j=1}^{k_i} \frac{\Lambda_{ij} + \delta_{ij}{}^{2} r_{ij}}{\left(\Lambda_{ij} + r_{ij} \right)^{2}} \quad (15)$$

Where δ_{ij} is the *jth* element of $\delta_{ij} = \psi_i{}' \hat{\beta}_R$ and r_{ij} is non-stochastic

Minimizing the TMSE in (15) with respect to r_{ij} gives us Firinguetti (1997). The following present some ridge parameters.

$$R_{SF} = r_{ij} = \frac{1}{\left[\hat{\alpha}^{+}_{SUR}(R) \right]^{2}} \quad (16)$$

The median of r_{ij} proposed by Kibria for single equation version while Alkhamisi and Shukur developed it for SUR model

$$R_{SK} = median\left(\frac{1}{\left[\hat{\alpha}^{+}_{SUR}(R)\right]^{2}}\right) \tag{17}$$

The maximum of r_{ij} proposed by Alkhamisi and Shukur

$$R_{AS} = \max\left(\frac{1}{\left[\hat{\alpha}^{+}_{SUR}(R)\right]^{2}}\right) \tag{18}$$

The modified Hoerl and Kennard (1970) for SUR ridge parameter is given by

$$R_{MSHK} = \left(\frac{1}{\max_{ij}\left(\left(\hat{\alpha}^{+}_{SUR}(R)\right)^{2}\right)}\right)^{1/\bar{k}} \tag{17}$$

The modified (Hoerl et al., 1975) harmonic mean for SUR is given by

$$R_{MSham} = \left(\frac{n}{\sum_{i=1}^{M}\sum_{j=1}^{n_i}\frac{1}{r_{ij}}}\right)^{1/\bar{k}} = \left(\frac{n}{\sum_{i=1}^{M}\sum_{j=1}^{n_i}\left(\hat{\alpha}^{+}_{SUR}(R)\right)^{2}}\right)^{1/\bar{k}} \tag{18}$$

The modified Kibria (2003) of the arithmetic mean extended to SUR is given by

$$R_{MSarith} = \left(\frac{1}{n}\sum_{i=1}^{M}\sum_{j=1}^{n_i}\frac{1}{\left(\hat{\alpha}^{+}_{SUR}(R)\right)^{2}}\right)^{1/\bar{k}} \tag{19}$$

3. THE SIMULATION STUDY

The Monte Carlo experiment was performed by generating data according to the following algorithm.

1. Generate the explanatory variable from $MVN_3(0, \Sigma_x)$
2. Set the true values of β to $(1,1,1,1)'$.
3. The variance-covariance is given as

$$\Sigma = \begin{bmatrix} \sigma_{11}^2 & \sigma_{12}^2 & \sigma_{13}^2 \\ \sigma_{21}^2 & \sigma_{22}^2 & \sigma_{23}^2 \\ \sigma_{31}^2 & \sigma_{32}^2 & \sigma_{33}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0.6 & 0.8 \\ & 1 & 0.7 \\ & & 1 \end{bmatrix} \text{ such that } \begin{matrix} Y_1 : N(X_1\beta_1, \sigma_{11}^2) \\ Y_2 : N(X_2\beta_2, \sigma_{22}^2) \\ Y_3 : N(X_3\beta_3, \sigma_{33}^2) \end{matrix}$$

4. Simulate the vector random error from $MVN_3(0, \Sigma_e)$.
5. For a given X structure, transform the original model to the canonical form.
6. Compute the trace mean squared error of R_{SF}, R_{SK}, R_{AS}, R_{MSHK}, R_{MSham}, $R_{Msarith}$
7. Repeat the above step 10000 times.

Table 1: **Description of Variables, Equations, Observations and Contemporaneous Correlations**

Factors	Symbol	Design
Number of equations	M	3
Number of observations	T	20, 30, 50, 100
Correlation among the explanatory variables	ρ_x	0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9
Contemporaneous correlation between corresponding errors among the equations	ρ_Σ	0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9

4. RESULTS AND DISCUSSION

The simulation results are presented in Tables 2 to 10 for the various sample sizes considered.

Table 2: Estimation of the TMSE when ρ_{ε_M} = 0.1

n = 20						
$\rho_{x_i x_j}$	R_{SF}	R_{SK}	R_{AS}	R_{MSHK}	R_{MSham}	$R_{Msarith}$
0.0	2.7241	**1.7030**	2.6501	1.9460	1.8334	1.8572
0.1	3.1561	**1.8221**	3.1447	2.2233	2.1964	2.2637
0.2	3.3144	**2.0115**	3.2543	2.4883	2.4091	2.3299
0.3	3.4995	**2.2528**	3.0309	2.8419	2.6977	2.6245
0.4	3.9781	**2.5799**	2.9939	3.2906	3.0428	3.0097
0.5	4.3153	3.0454	3.0607	3.8883	3.8178	3.5221
0.6	5.2073	3.7515	2.9961	4.8853	4.4174	4.2615
0.7	6.1443	4.9313	**2.9305**	6.4952	5.7382	5.4936
0.8	8.1122	7.2989	**2.6495**	9.6244	8.6795	7.7465
0.9	12.9085	14.4294	**2.3872**	19.2340	16.7374	13.1959
n = 30						
0.0	1.7528	**0.8796**	2.5596	0.9430	0.9431	0.9295
0.1	2.0157	**1.0861**	2.4676	1.1704	1.1718	1.1512
0.2	1.8866	**1.1240**	2.4323	1.1885	1.1708	1.1691
0.3	2.0538	**1.2734**	2.3339	1.3422	1.3459	1.3235
0.4	2.2228	**1.4723**	2.2162	1.5626	1.5230	1.5296
0.5	2.4778	**1.7500**	2.1643	1.8636	1.7621	1.8167
0.6	2.8353	2.1651	**1.9944**	2.3228	2.3062	2.2515
0.7	3.4228	2.8533	**1.8818**	3.0723	2.7792	2.9612
0.8	4.6396	4.2269	**2.0488**	4.5845	4.0928	4.3592
0.9	7.9681	8.3458	**2.3603**	9.1220	7.6230	8.3378
n = 50						
0.0	1.3079	**0.5258**	2.3914	0.5522	0.5496	0.5496
0.1	1.5105	**0.5690**	2.6600	0.6078	0.6121	0.5496
0.2	1.5622	**0.6310**	2.5805	0.6743	0.6772	0.6714
0.3	1.6376	**0.7117**	2.3886	0.7656	0.7662	0.7596
0.4	1.8427	**0.8206**	2.3478	0.8870	0.8763	0.8767
0.5	1.9115	**0.9738**	2.1802	1.0565	1.0367	1.0427
0.6	2.1206	**1.2036**	1.9982	1.3029	1.2923	1.3208
0.7	2.4691	**1.5859**	1.7153	1.7344	1.7105	1.7071
0.8	3.1304	2.3457	**1.6242**	2.5817	2.5336	2.5264
0.9	5.0140	4.5955	**1.6375**	5.1793	4.6155	4.9130
n = 100						

$\rho_{x_i x_j}$	R_{SF}	R_{SK}	R_{AS}	R_{MSHK}	R_{MSham}	$R_{Msarith}$
0.0	1.0409	**0.2493**	2.5122	0.2624	0.2624	0.2619
0.1	1.1115	**0.2805**	2.4081	0.2946	0.2948	0.2944
0.2	1.1762	**0.3117**	2.3587	0.3296	0.3287	0.3276
0.3	1.2439	**0.3519**	2.1349	0.3716	0.3716	0.3708
0.4	1.2842	**0.4061**	2.1509	0.4267	0.4303	0.4288
0.5	1.4283	**0.4824**	1.9535	0.5123	0.5096	0.5100
0.6	1.6017	**0.5975**	1.7327	0.6379	0.6331	0.6324
0.7	1.7786	**0.7899**	1.8517	0.8387	0.8386	0.8358
0.8	2.2024	**1.1749**	1.5103	1.2559	1.2171	1.2416
0.9	3.2915	2.3221	**1.4843**	2.4965	2.2688	2.4354

Table 3: Estimation of the TMSE when ρ_{ε_M} = 0.2

n = 20						
$\rho_{x_i x_j}$	R_{SF}	R_{SK}	R_{AS}	R_{MSHK}	R_{MSham}	$R_{Msarith}$
0.0	2.7241	**1.7030**	2.6501	1.9460	1.8334	1.8572
0.1	3.1561	**1.8221**	3.1447	2.2233	2.1964	2.2637
0.2	3.3144	**2.0115**	3.2543	2.4883	2.4091	2.3299
0.3	3.4995	**2.2528**	3.0309	2.8419	2.6977	2.6245
0.4	3.9781	**2.5799**	2.9939	3.2906	3.0428	3.0097
0.5	4.3153	**3.0454**	3.0607	3.8883	3.8178	3.5221
0.6	5.2073	**3.7515**	2.9961	4.8853	4.4174	4.2615
0.7	6.1443	4.9313	**2.9305**	6.4952	5.7382	5.4936
0.8	8.1122	7.2989	**2.6495**	9.6244	8.6795	7.7465
0.9	12.9085	14.4294	**2.3872**	19.2340	16.7374	13.1959
n = 30						
0.0	1.7528	**0.8796**	2.5596	0.9430	0.9431	0.9295
0.1	2.0157	**1.0861**	2.4676	1.1704	1.1718	1.1512
0.2	1.8866	**1.1240**	2.4323	1.1885	1.1708	1.1691
0.3	2.0538	**1.2734**	2.3339	1.3422	1.3459	1.3235
0.4	2.2228	**1.4723**	2.2162	1.5626	1.5230	1.5296
0.5	2.4778	**1.7500**	2.1643	1.8636	1.7621	1.8167
0.6	2.8353	2.1651	**1.9944**	2.3228	2.3062	2.2515
0.7	3.4228	2.8533	**1.8818**	3.0723	2.7792	2.9612
0.8	4.6396	4.2269	**2.0488**	4.5845	4.0928	4.3592
0.9	7.9681	8.3458	**2.3603**	9.1220	7.6230	8.3378
n = 50						

$\rho_{x_i x_j}$	R_{SF}	R_{SK}	R_{AS}	R_{MSHK}	R_{MSham}	$R_{Msarith}$
0.0	1.3079	**0.5258**	2.3914	0.5522	0.5496	0.5496
0.1	1.5105	0.5690	2.6600	0.6078	0.6121	**0.5496**
0.2	1.5622	**0.6310**	2.5805	0.6743	0.6772	0.6714
0.3	1.6376	**0.7117**	2.3886	0.7656	0.7662	0.7596
0.4	1.8427	**0.8206**	2.3478	0.8870	0.8763	0.8767
0.5	1.9115	**0.9738**	2.1802	1.0565	1.0367	1.0427
0.6	2.1206	**1.2036**	1.9982	1.3029	1.2923	1.3208
0.7	2.4691	**1.5859**	1.7153	1.7344	1.7105	1.7071
0.8	3.1304	2.3457	**1.6242**	2.5817	2.5336	2.5264
0.9	5.0140	4.5955	**1.6375**	5.1793	4.6155	4.9130
n = 100						
0.0	1.0409	**0.2493**	2.5122	0.2624	0.2624	0.2619
0.1	1.1115	**0.2805**	2.4081	0.2946	0.2948	0.2944
0.2	1.1762	**0.3117**	2.3587	0.3296	0.3287	0.3276
0.3	1.2439	**0.3519**	2.1349	0.3716	0.3716	0.3708
0.4	1.2842	**0.4061**	2.1509	0.4267	0.4303	0.4288
0.5	1.4283	**0.4824**	1.9535	0.5123	0.5096	0.5100
0.6	1.6017	**0.5975**	1.7327	0.6379	0.6331	0.6324
0.7	1.7786	**0.7899**	1.8517	0.8387	0.8386	0.8358
0.8	2.2024	**1.1749**	1.5103	1.2559	1.2171	1.2416
0.9	3.2915	2.3221	**1.4843**	2.4965	2.2688	2.4354

Table 4: Estimation of the TMSE when ρ_{ε_M} = 0.3

n = 20						
$\rho_{x_i x_j}$	R_{SF}	R_{SK}	R_{AS}	R_{MSHK}	R_{MSham}	$R_{Msarith}$
0.0	4.4364	**3.1585**	3.4191	4.0923	3.7694	3.5484
0.1	4.6985	**3.2704**	3.6459	4.5523	4.0727	3.8268
0.2	4.8443	**3.5862**	3.7663	5.0058	4.6849	4.1974
0.3	5.1843	4.0025	**3.5316**	5.7578	5.2312	4.6904
0.4	6.0958	4.5636	**3.5370**	6.6831	5.7888	5.2634
0.5	6.4806	5.3527	**3.4249**	7.9565	6.9319	6.0409
0.6	7.8591	6.5406	**3.2512**	9.8495	8.4989	7.1751
0.7	9.6718	8.5386	**3.1182**	13.1986	10.9684	8.7965
0.8	12.9665	12.5625	**3.0428**	19.7149	15.9481	11.5669
0.9	20.1303	24.7207	**2.6934**	39.1856	31.5178	17.3482
n = 30						

$\rho_{x_i x_j}$	R_{SF}	R_{SK}	R_{AS}	R_{MSHK}	R_{MSham}	$R_{Msarith}$
0.0	2.5559	**1.4774**	3.1227	1.7088	1.6529	1.6322
0.1	3.2819	**1.9434**	3.0202	2.3068	2.2406	2.1774
0.2	2.8748	**1.8663**	3.1471	2.1246	1.9880	2.0237
0.3	3.2181	**2.0914**	3.0108	2.3745	2.3561	2.2689
0.4	3.4118	**2.4072**	2.8903	2.7668	2.6180	2.6009
0.5	3.8312	2.8589	**2.7930**	3.2910	3.0633	3.0665
0.6	4.2422	3.5413	**2.6653**	4.0799	3.7792	3.7623
0.7	5.1784	4.6769	**2.5512**	5.4039	4.6203	4.9021
0.8	7.0261	6.9158	**2.4457**	8.0539	6.5339	7.0715
0.9	12.3263	13.4249	**3.1916**	16.0021	12.9488	12.9724
n = 50						
0.0	2.0839	**0.8765**	3.2458	0.9842	0.9751	0.9675
0.1	2.2180	**0.9320**	3.4115	1.0548	1.0294	1.0332
0.2	2.3136	**1.0212**	3.1652	1.1773	1.1616	1.1386
0.3	2.3995	**1.1424**	3.0954	1.3216	1.2925	1.2808
0.4	2.6592	**1.3061**	2.9013	1.5121	1.4838	1.4695
0.5	2.8170	**1.5392**	2.6045	1.7835	1.7827	1.7374
0.6	3.1719	**1.8926**	2.4989	2.2059	2.1896	2.1408
0.7	3.7389	2.4843	**2.4472**	2.9244	2.7646	2.7925
0.8	4.6973	3.6591	**2.2144**	4.3684	4.0618	4.0599
0.9	7.3911	7.1374	**1.8119**	8.6735	7.3180	7.6621
n = 100						
0.0	1.5262	**0.4097**	3.2237	0.4527	0.4535	0.4515
0.1	1.5548	**0.4540**	3.0327	0.4982	0.4996	0.4974
0.2	1.6194	**0.5003**	3.0031	0.5500	0.5510	0.5489
0.3	1.7404	0.5613	2.6777	0.6213	0.6165	0.6172
0.4	1.9215	**0.5932**	3.0337	0.6804	0.6791	0.6769
0.5	1.9974	**0.7621**	2.3815	0.8490	0.8439	0.8407
0.6	2.2374	**0.9399**	2.3719	1.0477	1.0376	1.0370
0.7	2.4568	**1.2373**	2.1602	1.3809	1.3631	1.3639
0.8	3.0012	**1.8283**	1.9015	2.0697	1.9119	2.0118
0.9	4.6149	3.5725	**1.6543**	4.1119	3.2950	3.8887

Table 5: Estimation of the TMSE when ρ_{ε_M} = 0.4

n = 20						
$\rho_{x_i x_j}$	R_{SF}	R_{SK}	R_{AS}	R_{MSHK}	R_{MSham}	$R_{Msarith}$
0.0	5.2426	4.0827	**3.4019**	5.4604	4.9497	4.5535
0.1	5.5365	4.1841	**3.6751**	5.9927	5.2997	4.8593
0.2	5.7043	4.5650	**3.7694**	6.7307	5.6361	5.2674
0.3	6.0359	5.0780	**3.6380**	7.5464	6.7416	5.8387
0.4	7.1788	5.7743	**3.5112**	8.7812	7.4884	6.4863
0.5	7.4854	6.7551	**3.2994**	10.4224	9.0645	7.4247
0.6	9.1126	8.2362	**3.4059**	13.0595	10.7611	8.6049
0.7	11.4381	10.7320	**3.1926**	17.2963	14.0909	10.5087
0.8	15.3927	15.7612	**2.9645**	25.6946	21.2295	13.5146
0.9	23.7672	30.9871	**2.7361**	51.3022	41.0121	19.2689
n = 30						
0.0	2.9318	**1.8128**	3.3896	2.0995	2.0681	2.0054
0.1	3.1403	**2.0435**	3.3214	2.3562	2.1289	2.2322
0.2	4.0963	**2.5067**	3.3006	2.8970	2.8453	2.7249
0.3	3.6239	**2.5067**	3.3006	2.8970	2.8453	2.7249
0.4	3.8228	**2.8862**	2.8891	3.3705	3.0446	3.1262
0.5	4.2875	3.4203	**2.7543**	3.9538	3.8652	3.6757
0.6	4.8278	4.2296	**2.5244**	4.9586	4.4601	4.5049
0.7	5.9245	5.5671	**2.6560**	6.5562	5.7343	5.5058
0.8	8.0933	8.1946	**2.7404**	9.7694	8.0205	8.2776
0.9	14.2688	15.7821	**3.3901**	19.3131	16.1297	14.8551
n = 50						
0.0	2.2661	**1.0606**	3.4731	1.2093	1.1909	1.1818
0.1	2.4396	**1.1135**	3.5010	1.2770	1.2073	1.2438
0.2	2.5134	**1.2129**	3.0616	1.4023	1.3934	1.3662
0.3	2.6284	**1.3515**	3.0135	1.5719	1.5561	1.5307
0.4	2.9362	**1.5426**	3.1322	1.8158	1.7368	1.7468
0.5	3.1499	**1.8128**	2.8992	2.1536	2.0759	2.0634
0.6	3.5619	**2.2254**	2.5832	2.6455	2.5246	2.5276
0.7	4.1923	2.9139	**2.2861**	3.5109	3.2709	3.3074
0.8	5.3276	4.2851	**2.2721**	5.2177	4.7246	4.7908
0.9	8.4934	8.3637	**2.1079**	10.3903	8.5238	8.8009
n = 100						

$\rho_{x_i x_j}$	R_{SF}	R_{SK}	R_{AS}	R_{MSHK}	R_{MSham}	$R_{Msarith}$
0.0	1.7090	**0.4932**	3.4635	0.5524	0.5502	0.5493
0.1	1.6935	**0.5399**	3.1444	0.6006	0.5987	0.5973
0.2	1.7563	**0.5920**	2.9637	0.6561	0.6598	0.6561
0.3	1.8744	**0.6619**	2.8396	0.7415	0.7331	0.7341
0.4	1.9639	**0.7579**	2.6505	0.8679	0.8454	0.8430
0.5	2.1703	**0.8951**	2.6939	1.0185	1.0016	0.9956
0.6	2.4264	**1.1022**	2.4059	1.2418	1.2251	1.2284
0.7	2.6614	**1.4477**	2.2379	1.6430	1.6048	1.6116
0.8	3.2735	2.1345	**1.6764**	2.4587	2.1372	2.3782
0.9	5.1422	4.1593	**1.5697**	4.8789	3.8658	4.5681

Table 6: Estimation of the TMSE when ρ_{ε_M} = 0.5

n = 20						
$\rho_{x_i x_j}$	R_{SF}	R_{SK}	R_{AS}	R_{MSHK}	R_{MSham}	$R_{Msarith}$
0.0	6.0517	5.0694	**3.4944**	7.1146	5.8352	5.5843
0.1	6.4025	5.1743	**3.7265**	7.6866	6.5889	5.8948
0.2	6.5937	5.6299	**3.9027**	8.5370	7.2682	6.3527
0.3	6.8887	6.2262	**3.6904**	9.6027	8.1552	6.9600
0.4	8.3452	7.0523	**3.6298**	11.1054	9.4363	7.6539
0.5	8.6036	8.2269	**3.3644**	13.1434	11.3115	8.7334
0.6	10.4139	10.0264	**3.3901**	16.3152	13.4945	10.1027
0.7	13.2295	13.0443	**3.3216**	21.7377	17.4181	12.0116
0.8	17.9729	19.1319	**3.0771**	31.9817	26.9792	15.1422
0.9	27.6136	37.5916	**2.8111**	64.4182	52.0031	20.7796
n = 30						
0.0	3.2697	**2.1543**	3.3123	2.5388	2.4654	2.3966
0.1	3.4547	**2.3929**	3.2169	2.7895	2.4761	2.6232
0.2	3.5959	**2.6088**	3.5648	3.0647	2.6627	2.8361
0.3	3.9792	**2.9135**	3.1933	3.3924	3.3349	3.1801
0.4	4.2174	3.3462	**2.8979**	3.9589	3.4521	3.6199
0.5	4.7503	3.9619	**3.0001**	4.6901	4.1051	4.2334
0.6	5.3648	4.8871	**2.7113**	5.8110	4.9986	5.1675
0.7	6.6197	6.4175	**2.6798**	7.6809	6.5188	6.6625
0.8	9.0915	9.4168	**2.8492**	11.4335	9.3806	9.4510
0.9	16.1123	18.0697	**3.5804**	22.6102	18.3181	16.6074
n = 50						

$\rho_{x_i x_j}$	R_{SF}	R_{SK}	R_{AS}	R_{MSHK}	R_{MSham}	$R_{Msarith}$
0.0	2.5119	**1.2524**	3.7271	1.4303	1.4257	1.3979
0.1	2.5914	**1.2968**	3.3482	1.4980	1.4326	1.4549
0.2	2.7407	**1.4068**	3.3381	1.6427	1.5759	1.5864
0.3	2.8599	**1.5627**	3.2575	1.8268	1.8221	1.7756
0.4	3.1859	**1.7780**	3.0814	2.1085	2.0495	2.0233
0.5	3.4362	**2.0865**	2.9229	2.5110	2.2474	2.3899
0.6	3.9149	**2.5547**	2.7926	3.0971	2.8767	2.9073
0.7	4.6272	3.3399	**2.4509**	4.0819	3.7157	3.7874
0.8	5.9022	4.9093	**2.3574**	6.0465	5.3514	5.4519
0.9	9.4964	9.5983	**2.2567**	12.0793	9.6970	9.8773
n = 100						
0.0	1.8196	**0.5783**	3.5273	0.6509	0.6466	0.6460
0.1	1.8404	**0.6239**	3.2795	0.7002	0.6955	0.6942
0.2	1.8689	**0.6820**	3.2119	0.7638	0.7639	0.7595
0.3	1.9928	**0.7602**	2.8805	0.8588	0.8369	0.8478
0.4	2.0958	**0.8679**	2.8339	0.9809	0.9769	0.9720
0.5	2.3152	**1.0237**	2.6791	1.1661	1.1091	1.1466
0.6	2.6029	**1.2586**	2.4123	1.4367	1.4055	1.4126
0.7	2.8752	**1.6508**	2.2181	1.8875	1.8155	1.8508
0.8	3.5348	2.4308	**1.7978**	2.8319	2.4379	2.7162
0.9	5.6228	4.7274	**1.6829**	5.6189	4.3238	5.1829

Table 7: Estimation of the TMSE when ρ_{ε_M} = 0.6

n = 20						
$\rho_{x_i x_j}$	R_{SF}	R_{SK}	R_{AS}	R_{MSHK}	R_{MSham}	$R_{Msarith}$
0.0	6.9306	6.1561	**3.8124**	8.7752	7.5949	6.5467
0.1	7.1789	6.2171	**3.6924**	9.5479	7.8399	7.0107
0.2	7.4311	6.7129	**3.9301**	10.4090	8.8784	7.4420
0.3	7.6854	7.3875	**3.7236**	11.7216	10.0774	8.1061
0.4	9.5166	8.3099	**3.6638**	13.5182	11.5133	8.8254
0.5	9.6629	9.6889	**3.4732**	15.9931	13.3540	10.0047
0.6	11.7057	11.7832	**3.4202**	19.8294	16.2574	11.4809
0.7	15.0172	15.3241	**3.2569**	26.3569	21.1958	13.5587
0.8	20.5078	22.4420	**3.0725**	38.8125	32.5859	16.7602
0.9	31.4117	44.0569	**2.8295**	79.4982	64.0343	22.2808
n = 30						
0.0	3.6664	**2.5074**	3.5588	2.9698	2.8454	2.7664
0.1	3.7913	**2.7291**	3.4927	3.2103	2.8192	2.9886
0.2	3.8995	**2.9664**	3.5088	3.5118	3.1565	3.2308
0.3	4.3172	3.3033	**3.1799**	3.8906	3.7732	3.5944
0.4	4.5887	3.7879	**3.0086**	4.5249	3.8976	4.1037
0.5	5.1858	4.4789	**2.9724**	5.3597	4.5480	4.7823

0.6	5.8820	5.5167	**2.8319**	6.6296	5.6417	7.0024
0.7	7.2793	7.2294	**2.7626**	8.7508	7.3918	7.4373
0.8	10.0287	10.5886	**2.9940**	13.0389	10.4659	10.4809
0.9	17.8416	20.2743	**3.6980**	25.8175	20.5886	18.2370
n = 50						
0.0	2.6623	**1.4393**	3.5769	1.6661	1.5721	1.6113
0.1	2.7302	**1.4759**	3.1716	1.7104	1.6343	1.6605
0.2	2.9459	**1.5965**	3.6046	1.8722	1.7822	1.7958
0.3	3.0584	**1.7673**	3.1269	2.0818	2.0521	2.0072
0.4	3.3971	**2.0063**	3.0272	2.4050	2.2533	2.2926
0.5	3.6942	**2.3520**	3.0337	2.8463	2.6526	2.6832
0.6	4.2248	2.8762	**2.8419**	3.5149	3.2678	3.2806
0.7	5.0436	3.7580	**2.6563**	4.6312	4.1351	4.2352
0.8	6.4559	5.5198	**2.5334**	6.9058	5.9249	6.0634
0.9	10.4224	10.8040	**2.1772**	13.6911	10.6485	10.9573
n = 100						
0.0	1.8796	**0.6621**	3.3685	0.7536	0.7460	0.7417
0.1	1.9409	**0.7061**	3.2819	0.7975	0.7847	0.7889
0.2	1.9702	**0.7690**	3.2816	0.8854	0.8634	0.8591
0.3	2.1055	**0.8549**	2.9984	0.9725	0.9419	0.9587
0.4	2.2079	**0.9747**	2.7997	1.1155	1.0976	1.0959
0.5	2.4426	**1.1469**	2.7159	1.3161	1.2622	1.2907
0.6	2.7643	**1.4092**	2.4839	1.6248	1.5089	1.5855
0.7	3.0755	**1.8466**	2.3239	2.1384	2.0449	2.0770
0.8	3.7906	2.7157	**1.8863**	3.2244	2.8613	3.0465
0.9	6.0884	5.2793	**1.7357**	6.3314	4.7076	5.7838

Table 8: Estimation of the TMSE when ρ_{ε_M} = 0.7

n = 20						
$\rho_{x_i x_j}$	R_{SF}	R_{SK}	R_{AS}	R_{MSHK}	R_{MSham}	$R_{Msarith}$
0.0	7.54445	7.0871	3.5942	**1.0572**	8.7233	7.6268
0.1	7.9482	7.3022	**3.8314**	11.3765	9.7336	7.9522
0.2	8.2992	7.7887	**3.6062**	12.5002	10.2077	8.5185
0.3	8.6672	8.5123	**3.7320**	13.9028	11.4204	9.1691
0.4	10.5744	9.5642	**3.5074**	16.0159	13.2361	10.1549
0.5	10.5932	11.1224	**3.6465**	18.6411	16.1658	11.0553
0.6	12.8141	13.5098	**3.4552**	23.4322	18.3584	12.5827
0.7	16.1601	17.5378	**3.4233**	30.8755	24.8734	14.6517
0.8	22.2237	25.6958	**3.3756**	45.5177	37.2322	17.7843
0.9	35.4320	50.4295	**3.2154**	89.7714	74.7612	23.0850
n = 30						
0.0	3.9246	**2.8548**	3.3914	3.4058	3.2073	3.1494
0.1	4.0430	**3.0553**	3.4377	3.6101	3.2995	3.3235

0.2	4.2084	**3.3172**	3.6483	3.9333	3.5761	3.5915
0.3	4.6760	3.6952	**3.5567**	4.3683	4.2186	3.9840
0.4	4.9887	4.2278	**3.2852**	5.0658	4.2710	4.5318
0.5	5.5554	4.9824	**3.2132**	5.9925	5.0877	5.2536
0.6	6.4217	6.1219	**3.0669**	7.4111	6.1181	6.3671
0.7	7.9879	8.0082	**2.8610**	9.7835	8.0685	8.1341
0.8	10.9770	11.7005	**2.9971**	14.5715	11.5462	11.4265
0.9	19.3626	22.4138	**3.7493**	28.7452	23.3051	19.6834
n = 50						
0.0	3.1109	**1.7291**	3.4547	2.0554	2.0046	1.9767
0.1	2.8893	**1.6489**	3.6715	1.9208	1.7786	1.8418
0.2	3.0472	**1.7781**	3.3798	2.0952	1.9477	2.0032
0.3	3.2502	**1.9676**	3.2511	2.3204	2.2728	2.2258
0.4	3.5032	**2.2319**	3.1312	2.6753	2.5431	2.5380
0.5	3.8589	**2.6133**	2.8339	3.1746	2.8682	2.9746
0.6	4.4246	3.1949	**2.7417**	3.9252	3.4878	3.6150
0.7	5.3462	4.1752	**2.6621**	5.1515	4.5237	4.6840
0.8	6.8862	6.1410	**2.4202**	7.6980	6.3978	6.6908
0.9	11.4828	12.0027	**2.2638**	15.2868	11.8962	11.9265
0.0	1.9486	**0.7434**	3.1681	0.8378	0.8364	0.8337
0.1	2.0159	**0.7852**	3.1119	0.8978	0.8799	0.8792
0.2	2.0464	**0.8491**	3.1900	0.9586	0.9623	0.9541
0.3	2.1855	**0.9420**	2.8275	1.0787	1.0463	1.0630
0.4	2.2639	**1.0742**	2.6176	1.2345	1.2115	1.2133
0.5	2.5129	**1.2639**	2.5441	1.4595	1.3803	1.4303
0.6	2.8330	**1.5524**	2.3566	1.8015	1.6451	1.7569
0.7	3.2555	**2.0343**	2.2793	2.3696	2.2521	2.2914
0.8	4.1314	2.9910	**2.1541**	3.5383	2.9249	3.3381
0.9	6.5555	5.8132	**1.9176**	7.0148	5.2127	6.3329

Table 9: Estimation of the TMSE when ρ_{ε_M} = 0.8

n = 20						
$\rho_{x_i x_j}$	R_{SF}	R_{SK}	R_{AS}	R_{MSHK}	R_{MSham}	$R_{Msarith}$
0.0	8.4844	8.1857	**3.7059**	12.1985	9.8219	8.4235
0.1	8.7051	8.3176	**3.8395**	13.0972	10.8404	8.8873
0.2	8.9599	8.8721	**3.8531**	14.2930	12.1128	9.3979
0.3	9.2023	9.7127	**3.7736**	15.8596	13.3683	10.1349
0.4	11.6107	10.8919	**3.7416**	18.1963	15.4063	10.8654
0.5	11.4290	12.5782	**3.6497**	21.4290	17.5233	12.1150
0.6	13.9653	15.2201	**3.6228**	26.4636	21.5541	13.6577

0.7	18.1661	19.7007	**3.3949**	34.9672	27.8295	15.8549
0.8	24.9430	28.7803	**3.2931**	52.0074	41.5708	19.2171
0.9	38.2256	56.4242	**3.0619**	103.1580	84.1153	24.4354
n = 30						
0.0	4.2870	**3.1720**	3.5958	3.8041	3.4281	3.4933
0.1	4.3824	**3.3553**	3.5710	3.9873	3.5114	3.6595
0.2	4.4531	3.6305	**3.5400**	4.3465	3.8657	3.9417
0.3	4.9265	4.0291	**3.3621**	4.8061	4.5332	4.3524
0.4	5.2659	4.6075	**3.2990**	5.5721	4.7319	4.9245
0.5	5.9658	5.4359	**3.0332**	6.5761	5.6910	5.7440
0.6	6.8151	6.6822	**2.9962**	8.1474	6.7541	6.9412
0.7	8.5012	8.7387	**3.0455**	10.7338	9.2579	8.8080
0.8	11.7561	12.7684	**3.1143**	16.0134	12.6444	12.3517
0.9	21.0367	24.4110	**3.9532**	31.6772	25.4816	21.0404
n = 50						
0.0	2.9422	**1.7907**	3.6842	2.0957	1.9523	1.9988
0.1	3.0408	**1.8077**	3.4764	2.1110	1.9913	2.0313
0.2	3.5361	**2.2578**	3.2391	2.7155	2.5213	2.5721
0.3	3.4126	**2.1521**	3.3105	2.5493	2.4584	2.4409
0.4	3.7539	**2.4395**	3.0644	2.9450	2.6665	2.7733
0.5	4.1201	**2.8561**	3.1099	3.4807	3.1483	3.2375
0.6	4.7738	3.4884	**2.9578**	4.3058	3.7739	3.9370
0.7	5.7613	4.5518	**2.6919**	5.6606	4.9281	5.0938
0.8	7.4372	6.6843	**2.5348**	8.4228	7.0257	7.2290
0.9	12.1693	13.0962	**2.2042**	16.6984	13.0281	12.7529
n = 100						
0.0	2.0175	**0.8191**	3.5530	0.9363	0.9248	0.9191
0.1	2.1164	**0.8609**	3.2894	0.9763	0.9608	0.9639
0.2	2.1321	**0.9325**	3.2823	1.0616	1.0489	1.0453
0.3	2.2938	**1.0336**	3.0477	1.1832	1.1301	1.1637
0.4	2.4083	**1.1754**	2.9661	1.3501	1.3168	1.3257
0.5	2.6687	**1.3806**	2.6937	1.5999	1.4913	1.5611
0.6	3.0409	**1.6938**	2.6531	1.9719	1.8092	1.9144
0.7	3.4217	**2.2180**	2.3014	2.5873	2.4256	2.5011
0.8	4.2608	3.2577	**2.1162**	3.8666	3.1651	3.6535
0.9	6.9589	6.3305	**1.8282**	7.6666	5.6216	6.8901

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Table 10: Estimation of the TMSE when ρ_{ε_M} = 0.9

n = 20						
$\rho_{x_i x_j}$	R_{SF}	R_{SK}	R_{AS}	R_{MSHK}	R_{MSham}	$R_{Msarith}$
0.0	8.8863	8.9032	**3.7227**	13.5046	11.7777	9.2903
0.1	9.3588	9.2383	**3.7898**	14.6516	12.2453	9.6343
0.2	9.6775	9.7680	**3.8111**	15.9710	12.8461	10.8621
0.3	10.0474	10.6448	**3.7855**	17.7040	14.6349	10.8621
0.4	1.2503	1.1912	3.6902	2.0082	1.6882	1.1873
0.5	12.2781	13.7870	**3.7451**	23.6885	20.1801	12.9151
0.6	14.8775	16.6886	**3.6454**	29.2025	24.2477	14.5427
0.7	19.0062	21.6053	**3.4998**	38.5737	31.3474	16.8167
0.8	26.2383	31.5481	**3.5191**	57.0241	47.4617	19.9584
0.9	41.8226	61.7549	**3.3189**	114.8547	89.8884	25.1641
n = 30						
0.0	4.4475	**3.4791**	3.5450	4.1695	3.7942	3.8041
0.1	4.5483	3.6465	**3.4481**	4.3405	3.7732	3.9583
0.2	4.7283	3.9410	**3.7489**	4.7189	4.2369	4.2344
0.3	5.2634	4.3826	**3.6323**	5.2298	4.9803	4.6746
0.4	5.6194	5.0054	**3.3936**	6.0532	5.0585	5.3129
0.5	6.2993	5.8889	**3.4836**	7.1546	5.9996	6.1309
0.6	7.3017	7.2222	**3.1993**	8.8429	7.5233	7.4121
0.7	9.1230	9.4353	**3.0594**	11.6468	9.6868	9.4701
0.8	12.6058	13.7717	**3.2420**	17.3500	13.9689	13.1220
0.9	2.2395	**2.6394**	3.9494	3.4291	2.7532	2.2250
n = 50						
0.0	3.1181	**1.9482**	3.5726	2.2677	2.1578	2.1701
0.1	3.1484	**1.9630**	3.6461	2.2941	2.1459	2.1926
0.2	3.3492	**2.1121**	3.4636	2.4767	2.4277	2.3731
0.3	3.5730	**2.3352**	3.3127	2.7635	2.6956	2.6272
0.4	3.8533	**2.6473**	3.2851	3.1921	2.8388	2.9871
0.5	4.2603	3.0958	**2.9194**	3.7723	3.3572	3.4997
0.6	4.9336	3.7809	**2.7816**	4.6510	4.2633	4.2565
0.7	6.0025	4.9396	**2.7212**	6.1267	5.3855	5.4670
0.8	7.7683	7.2599	**2.4954**	9.1369	7.2413	7.7733
0.9	13.1144	14.1905	**2.4475**	18.1517	13.7851	13.5676
n = 100						

$\rho_{x_i x_j}$	R_{SF}	R_{SK}	R_{AS}	R_{MSHK}	R_{MSham}	$R_{Msarith}$
0.0	2.1128	**0.8904**	3.5402	1.0068	1.0172	0.9987
0.1	2.1791	**0.9311**	3.2981	1.0710	1.0363	1.0449
0.2	2.1953	**1.0060**	3.0604	1.1480	1.1363	1.1321
0.3	2.3387	**1.1134**	2.8275	1.2781	1.2514	1.2582
0.4	2.4604	**1.2661**	2.7082	1.4629	1.3966	1.4342
0.5	2.7258	**1.4893**	2.6404	1.7302	1.6071	1.6856
0.6	3.0868	**1.8271**	2.3821	2.1368	1.9164	2.0717
0.7	3.5837	2.3911	**2.3266**	2.8116	2.5598	2.6941
0.8	4.5831	3.5138	**2.2831**	4.1911	3.3405	3.9139
0.9	7.3918	6.8330	**2.0181**	8.2959	6.0953	7.3643

5. DISCUSSION OF RESULTS

The results of Tables 2 and 3 reveal that the TMSE values for all the estimators decrease as the sample sizes increase when considering the different correlation levels among the explanatory variables. When n = 20, 30, ρ_{ε_M}= 0.1, 0.2, the estimator R_{SK} has the lowest TMSE value and the best performance in terms of the TMSE criterion compared to the remaining estimators for $\boldsymbol{\rho}_{x_i x_j}$ = 0.0, 0.1, 0.2, 0.3, 0.4, 0.5, while the estimator R_{AS} has the lowest TMSE value and the best performance in terms of the TMSE criterion compared to the remaining estimators for $\boldsymbol{\rho}_{x_i x_j}$ = 0.6, 0.7, 0.8, 0.9. However, as the sample size increases, the estimator R_{SK} performance becomes more dominant over the other estimators. When n = 50, ρ_{ε_M}= 0.1, 0.2, the estimator R_{SK} has the best performance in terms of the TMSE criterion compared to the remaining estimators for $\boldsymbol{\rho}_{x_i x_j}$ = 0.0, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, while the estimator R_{AS} has the best performance in terms of the TMSE criterion compared to the remaining estimators for $\boldsymbol{\rho}_{x_i x_j}$ = 0.8, 0.9. The estimator $R_{Msarith}$ has the best performance in terms of the TMSE criterion compared to the remaining estimators for $\boldsymbol{\rho}_{x_i x_j}$ = 0.1. When n = 100, ρ_{ε_M}= 0.1, 0.2, the estimator R_{SK} has the lowest TMSE value and the best performance in terms of the TMSE criterion compared to the remaining estimators for $\boldsymbol{\rho}_{x_i x_j}$ = 0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, while the estimator R_{AS} has the lowest TMSE value and the best performance in terms of the TMSE criterion compared to the remaining estimators for $\boldsymbol{\rho}_{x_i x_j}$ = 0.9.

Table 4 reveals that the TMSE values for all the estimators decrease as the sample sizes increase when considering the different correlation levels among the explanatory variables. When n = 20, ρ_{ε_M}= 0.3, the estimator R_{SK} has the best performance in terms of the TMSE criterion compared to the remaining estimators for $\boldsymbol{\rho}_{x_i x_j}$ = 0.0, 0.1, 0.2, while the estimator R_{AS} has the best performance in terms of the TMSE criterion compared to the remaining estimators for $\boldsymbol{\rho}_{x_i x_j}$ = 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9. When n = 30, ρ_{ε_M}= 0.3, the estimator R_{SK} has the best performance in terms of the TMSE criterion compared to the remaining estimators for $\boldsymbol{\rho}_{x_i x_j}$ = 0.0, 0.1, 0.2, 0.3, 0.4, while the estimator R_{AS} has the best performance in terms of the TMSE criterion compared to the remaining estimators for $\boldsymbol{\rho}_{x_i x_j}$ = 0.5, 0.6, 0.7, 0.8, 0.9. When n = 50, ρ_{ε_M}= 0.3, the estimator R_{SK} has the best performance in terms of the TMSE criterion compared to the remaining estimators for $\boldsymbol{\rho}_{x_i x_j}$ = 0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, while the estimator R_{AS} has the best performance in terms of the TMSE criterion compared to the remaining estimators for $\boldsymbol{\rho}_{x_i x_j}$ = 0.7, 0.8, 0.9. When n = 100, ρ_{ε_M}= 0.3, the results are similar to when n = 100, ρ_{ε_M}= 0.1, 0.2, with the estimator R_{SK} outperforming the other estimators in all cases except for $\boldsymbol{\rho}_{x_i x_j}$ = 0.9.

Table 5 reveals that the TMSE values for all the estimators decrease as the sample sizes increase when considering the different correlation levels among the explanatory variables. When n = 20, ρ_{ε_M}= 0.4, the estimator R_{AS} has the best performance in terms of the TMSE criterion compared to the remaining estimators for all the cases of $\boldsymbol{\rho}_{x_i x_j}$. When n = 30, ρ_{ε_M}= 0.4, the estimator R_{SK} has the best performance in terms of the TMSE criterion compared to the remaining estimators for $\boldsymbol{\rho}_{x_i x_j}$ = 0.0, 0.1, 0.2, 0.3, 0.4, while the estimator R_{AS} has the best performance in terms of the TMSE criterion compared to the remaining estimators for $\boldsymbol{\rho}_{x_i x_j}$ = 0.5, 0.6, 0.7, 0.8, 0.9. When n = 50, ρ_{ε_M}= 0.4, the estimator R_{SK} has the best performance in terms of the TMSE criterion compared to the remaining estimators for $\boldsymbol{\rho}_{x_i x_j}$ = 0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, while the estimator R_{AS} has the best performance in terms of the TMSE criterion compared to the remaining estimators for $\boldsymbol{\rho}_{x_i x_j}$ = 0.7, 0.8, 0.9. When n = 100, ρ_{ε_M}= 0.4, the estimator R_{SK} outperforms the other estimators in all cases except for $\boldsymbol{\rho}_{x_i x_j}$ = 0.8, 0.9.

Table 6 also reveals that the TMSE values for all the estimators decrease as the sample sizes increase when considering the different correlation levels among the explanatory variables. When n = 20, ρ_{ε_M}= 0.5, the estimator R_{AS} has the best performance in terms of the TMSE criterion compared to the remaining estimators for all the cases of $\boldsymbol{\rho}_{x_i x_j}$. When n = 30, ρ_{ε_M}= 0.5, the estimator R_{SK} has the best performance in terms of the TMSE criterion compared to the remaining estimators for $\boldsymbol{\rho}_{x_i x_j}$ = 0.0, 0.1, 0.2, 0.3, while the estimator R_{AS} has the best performance in terms of the TMSE criterion compared to the remaining estimators for $\boldsymbol{\rho}_{x_i x_j}$ = 0.4, 0.5, 0.6, 0.7, 0.8, 0.9. When n = 50, ρ_{ε_M}= 0.5, the estimator R_{SK} has the best performance in terms of the TMSE criterion compared to the remaining estimators for $\boldsymbol{\rho}_{x_i x_j}$ = 0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, while the estimator R_{AS} has the best performance in terms of the TMSE criterion compared to the remaining estimators for $\boldsymbol{\rho}_{x_i x_j}$ = 0.7, 0.8, 0.9. When n = 100, ρ_{ε_M}= 0.5, the estimator R_{SK} outperforms the other estimators in all cases except for $\boldsymbol{\rho}_{x_i x_j}$ = 0.8, 0.9.

Tables 7 and 8 revealed that the TMSE values for all the estimators decrease as the sample sizes increase when considering the different correlation levels among the explanatory variables. When n = 20, ρ_{ε_M}= 0.6, 0.7, the estimator R_{AS} has the best performance in terms of the TMSE criterion compared to the remaining estimators for all the cases of $\boldsymbol{\rho}_{x_i x_j}$. When n = 30, ρ_{ε_M}= 0.6, 0.7, the estimator R_{SK} has the best performance in terms of the TMSE criterion compared to the remaining estimators for $\boldsymbol{\rho}_{x_i x_j}$ = 0.0, 0.1, 0.2, while the estimator R_{AS} has the best performance in terms of the TMSE criterion compared to the remaining estimators for $\boldsymbol{\rho}_{x_i x_j}$ = 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9. When n = 50, ρ_{ε_M}= 0.6, 0.7, the estimator R_{SK} has the best performance in terms of the TMSE criterion compared to the remaining estimators for $\boldsymbol{\rho}_{x_i x_j}$ = 0.0, 0.1, 0.2, 0.3, 0.4, 0.5, while the estimator R_{AS} has the best performance in terms of the TMSE criterion compared to the remaining estimators for $\boldsymbol{\rho}_{x_i x_j}$ = 0.6, 0.7, 0.8, 0.9. When n = 100, ρ_{ε_M}= 0.6, 0.7, the estimator R_{SK} outperforms the other estimators in all cases except for $\boldsymbol{\rho}_{x_i x_j}$ = 0.8, 0.9.

Table 9 reveals that the TMSE values for all the estimators decrease as the sample sizes increase when considering the different correlation levels among the explanatory variables. When n = 20, ρ_{ε_M}= 0.8, the estimator R_{AS} has the best performance in terms of the TMSE criterion compared to the remaining estimators for all the cases of $\boldsymbol{\rho}_{x_i x_j}$. When n = 30, ρ_{ε_M}= 0.8, the estimator R_{SK} has the best performance in terms of the TMSE criterion compared to the remaining estimators for $\boldsymbol{\rho}_{x_i x_j}$ = 0.0, 0.1, while the estimator R_{AS} has the best performance in terms of the TMSE criterion compared to the remaining estimators for $\boldsymbol{\rho}_{x_i x_j}$ = 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9.

When n = 50, ρ_{ε_M}= 0.8, the estimator R_{SK} has the best performance in terms of the TMSE criterion compared to the remaining estimators for $\boldsymbol{\rho}_{x_i x_j}$ = 0.0, 0.1, 0.2, 0.3, 0.4, 0.5, while the estimator R_{AS} has the best performance in terms of the TMSE criterion compared to the remaining estimators for $\boldsymbol{\rho}_{x_i x_j}$ = 0.6, 0.7, 0.8, 0.9. When n = 100, ρ_{ε_M}= 0.8, the estimator R_{SK} outperforms the other estimators in all cases except for $\boldsymbol{\rho}_{x_i x_j}$ = 0.8, 0.9.

Table 10 revealed that the TMSE values for all the estimators decrease as the sample sizes increase when considering the different correlation levels among the explanatory variables. When n = 20, ρ_{ε_M}= 0.9, the estimator R_{AS} has the best performance in terms of the TMSE criterion compared to the remaining estimators for all the cases of $\boldsymbol{\rho}_{x_i x_j}$. When n = 30, ρ_{ε_M}= 0.9, the estimator R_{SK} has the best performance in terms of the TMSE criterion compared to the remaining estimators for $\boldsymbol{\rho}_{x_i x_j}$ = 0.0, 0.9, while the estimator R_{AS} has the best performance in terms of the TMSE criterion compared to the remaining estimators for $\boldsymbol{\rho}_{x_i x_j}$ = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8. When n = 50, ρ_{ε_M}= 0.9, the estimator R_{SK} has the best performance in terms of the TMSE criterion compared to the remaining estimators for $\boldsymbol{\rho}_{x_i x_j}$ = 0.0, 0.1, 0.2, 0.3, 0.4, while the estimator R_{AS} has the best performance in terms of the TMSE criterion compared to the remaining estimators for $\boldsymbol{\rho}_{x_i x_j}$ = 0.5, 0.6, 0.7, 0.8, 0.9. When n = 100, ρ_{ε_M}= 0.8, the estimator R_{SK} outperforms the other estimators in all cases except for $\boldsymbol{\rho}_{x_i x_j}$ = 0.7, 0.8, 0.9.

6. CONCLUSION

The efficiency of six different shrinkage estimators: Firinguetti (R_{SF}), Alkhamisi and Shukur-Median (R_{SK}), Alkhamisi and Shukur-Maximum (R_{AS}), Hoerl and Kennard-Maximum (R_{MSHK}), Hoerl *et al*-Harmonic mean (R_{MSham}), and Kibria $(R_{Msarith})$ on the SUR model was explored. A three-equation joint model was considered ith different correlation levels among the explanatory variables ($\boldsymbol{\rho}_{x_i x_j}$) and contemporaneous correlation levels (ρ_{ε_M}) among the equations. Samples sizes 20, 30, 50 and 100 replicated 10000 times in turn were considered for the simulation study. Results from the study revealed that the TMSE values for all the estimators decrease as the sample sizes increase when considering the different correlation levels among the explanatory variables. When n = 20, ρ_{ε_M}= 0.9, the estimator R_{AS} had the best performance in terms of the TMSE criterion compared to the remaining estimators for all the cases of $\boldsymbol{\rho}_{x_i x_j}$.

When n = 30, ρ_{ε_M}= 0.9, the estimator R_{SK} gave the best performance in terms of the TMSE criterion compared to the remaining estimators for $\boldsymbol{\rho}_{x_i x_j}$ = 0.0, 0.9, while the estimator R_{AS} had the best performance in terms of the TMSE criterion compared to the remaining estimators for $\boldsymbol{\rho}_{x_i x_j}$ = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8. When n = 50, ρ_{ε_M}= 0.9, the estimator R_{SK} had the best performance in terms of the TMSE criterion compared to the remaining estimators for $\boldsymbol{\rho}_{x_i x_j}$ = 0.0, 0.1, 0.2, 0.3, 0.4, while the estimator R_{AS} gave the best performance in terms of the TMSE criterion compared to the remaining estimators for $\boldsymbol{\rho}_{x_i x_j}$ = 0.5, 0.6, 0.7, 0.8, 0.9. When n = 100, ρ_{ε_M}= 0.8, the estimator R_{SK} outperformed the other estimators in all cases except for $\boldsymbol{\rho}_{x_i x_j}$ = 0.7, 0.8, 0.9. Conclusively, Alkhamisi and Shukur-Median (R_{SK}) outperformed other estimators, followed by Alkhamisi and Shukur-Maximum (R_{AS}),

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