Heat Flow and Entropy Monotonicity Approach to Derivation of Functional Inequalities

Abolarinwa, A.
Department of Physical Sciences
Landmark University
Omu-Aran, Kwara State, Nigeria
E-mail: abolarinwa.abimbola@lmu.edu.ng

ABSTRACT

Heat flows and entropy formulas are two powerful analytic tools widely used in various settings of mathematical analysis and scientific research. Heat flow monotonicity formulas have evolved in recent years as a powerful tool in deriving functional and geometric inequalities which are in turn useful in mathematical analysis and applications. This article discusses the proofs of multilinear Hölder's inequalities and Logarithmic Sobolev using through the heat flow method. Precisely, two entropy monotonicity formulas are constructed via the heat flow. These inequalities can be easily seen as the consequence of the ability of functionals involving powers of smooth solution to the heat equation to approach their extremal values as time grows infinitely.

Keywords - Entropy, Heat Flow, Sobolev inequality, Hölder inequality, Gaussian measure.

iSTEAMS Multidisciplinary Conference Proceedings Reference Format
Abolarinwa, A. (2019): Heat Flow and Entropy Monotonicity Approach to Derivation of Functional Inequalities.
Proceedings of the 22nd iSTEAMS Multidisciplinary SPRING Conference. Aurora Conference centre, Osogbo, Nigeria. 17th – 19th December, 2019. Pp 55-66. www.isteams.net/spring2019. DOI - https://doi.org/ 10.22624/AIMS/iSTEAMS-2019/V22N1P5

1. INTRODUCTION

Heat flow, though classical, has attracted more attention in proving functional inequalities [2, 4] since the work of Carlen, Loss and Lieb [6]. Their work [6] was motivated by statistical mechanical considerations, and was used to derive some sharp inequalities on the sphere. Entropy formulas were used in the 70's to prove some embedding and hypercontractive estimates [7, 8], consequently leading to Sobolev-type inequalities. Since the complete proof of Poincare conjecture by Perelman [12], many entropy monotonicity formulas have been proved [1, 2, 3, 11, 15, 16] and thereby leading to several functional inequalities such as Harnack and Logarithmic Sobolev inequalities. A motivation for this is that applications of these inequalities are ubiquitous in various fields such as PDEs, Euclidean analysis, harmonic analysis, information theory, optimal transport, kinetic theory and so on (cf. [9, 16]). This article proves two important inequalities in mathematical analysis and applications, namely; multilinear Hölder's inequalities and Logarithmic Sobolev. In order to achieve this, two entropy monotonicity formulas are constructed and their monotonicity proved by the method of the heat flow. The first monotonicity formula is combined with convolution and diffusion semigroup properties of the heat kernel to establish the proof of the multilinear Hölder inequalities. The second entropy formula is shown to be intimately related to the concavity of the power of Shannon entropy and Fisher Information [13, 14], from which the associated logarithmic Sobolev inequality for probability measure in Euclidean setting is recovered. This approach, that is combining heat flow and entropy fomula, appeared more simplified than using either heat flow or entropy fomula independently.

1.1 Preliminaries

This article is concerned with the monotonicity formulas for the heat equation on Euclidean space, $\mathbb{R}^n$, and their geometric consequences for some functional inequalities, namely, Gaussian logarithmic Sobolev inequality introduced by L. Gross in [7] and the well known multilinear Hölder inequalities.

The heat equation

$$\left(\frac{\partial}{\partial t} - \Delta\right) u(t,x) = 0 \qquad (1.1)$$

with its positive solution $u = u(t,x)$, where $\frac{\partial}{\partial t}$ is the partial derivative with respect to t, $\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$ and $x = (x_1, .., x_n) \in \mathbb{R}^n$ is considered. The standard Gauss-Weierstrass heat kernel in $\mathbb{R}^n$ is given by $K(t,x,y) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4t}}$. It is well known that the convolution of K with $u(0,x) = u(x)$ solves the heat equation (1.1) and possesses important properties which include smoothness, positivity and diffusion semigroup. These properties are of paramount importance in this work. Now, set the solution of (1.1) to be a Gaussian density function with respect to the Lebesgue measure on $\mathbb{R}^n$,

$$u(t,x) := (4\pi\tau)^{-\frac{n}{2}} e^{-f(t,x)}, \quad \tau = \tau(t) > 0, \quad f: \mathbb{R}^n \to (0,\infty)$$

with normalization $\int_{\mathbb{R}^n} u d\mu = 1$ and $\frac{\partial \tau}{\partial t} = 1$, where $d\mu = (2\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{2}} dx$ is the Gaussian measure on $\mathbb{R}^n$ and dx is the Lebesque measure.

An entropy functional is defined as follows

$$W(\tau, f) = \int_{\mathbb{R}^n} (\tau|\nabla f|^2 + f - n)(4\pi\tau)^{-\frac{n}{2}} e^{-f} dx \qquad (1.2)$$

with $\int_{\mathbb{R}^n} (4\pi\tau)^{-\frac{n}{2}} e^{-f} dx = 1$.

This paper proves that entropy (1.2) is monotonically nonincreasing in time and shows that the resulting monotonicity formula is strongly related to classical entropies of Shannon and Fisher information. The entropies of Shannon and Fisher information for heat equation are respectively defined as:

$$H(u) = -\int_{\mathbb{R}^n} u \ln u \; dx \quad \text{and} \quad I(u) = \int_{\mathbb{R}^n} |\nabla \ln u|^2 u \; dx.$$

The entropy $H(u)$ was first introduced in Shannon [13]. They have numerous applications in information theory, combinatorics, thermodynamics and statistical mechanics. For instance $H(u)$ can be used to provide alternative proof of Loomis-Whitney inequality [10].

Note that Loomis-Whitney inequality is a natural generalization of multilinear Hölder inequality [5].

Proposition 1.1 *Let u be a positive solution to the heat equation (1.1). Then*

$$\frac{dH(u)}{dt} = I(u) \qquad (1.3)$$

and

$$\frac{d^2H(u)}{dt^2} = -2J(u) \qquad (1.4)$$

where $J(u) = \int_{\mathbb{R}^n} |\nabla_i \nabla_j \ln\ u|^2 u\ dx$ and ∇_i is the gradient operator on the i^{th} coordinate.

The proof of Proposition 1.1 is contained in Lemma 4.1 below. The relationship in the above proposition is known in literature as DeBruijin's identities and have been coupled to prove the concavity of Shannon entropy power. The concavity of entropy power can be viewed as being equivalent to the inequality

$$J(u) \geq \frac{1}{n} I^2, \quad \text{where} \quad -2J(u) = \frac{d}{dt} I(u). \qquad (1.5)$$

Various proofs of this inequality can be found in literature, e.g. [16] and [17].

In [16], Toscani considers scaling properties of Shannon entropy and Fisher Information respectively as:

$$H(u) = H(u) - n\ln a \quad \text{and} \quad I(u) = a^2 I(u)$$

using scaling factor $g(u_a) \to g_a(v) = a^2 g(av), \quad a > 0,$ which preserves the total mass of the function g. A more direct tool is employed to dilate these quantities and the scalings have direct consequences on the monotone property of the quantity W in (1.2). All these are discussed in Section (4).

2. STATEMENT OF RESULTS

The main results of this paper are stated in this section. The first entropy functional is constructed as follows

$$\Phi(t) = \int_{\mathbb{R}^n} \prod_{j=1}^m f_j(t,x)^{\frac{1}{p_j}} d\mu(x) \qquad (2.1)$$

for all nonnegative functions $f_j \in L^{p_j}(\mathbb{R}^n)$. Let f_j be a solution to the heat equation, the entropy (2.1) is known to be differentiable and smoothly continuous by the smoothing properties of the heat kernel semigroup for all $t > 0$. The monotonicity formula for $\Phi(t)$ is used to prove the following theorem.

Theorem 2.1 *Let the product of m-functions f_j be $\prod_{j=1}^m f_j(x),\ m \in \mathbb{N}$ and $1 \leq j \leq m$. For each $1 \leq p_j \leq \infty$, we have the inequality*

$$\int_{\mathbb{R}^n} \prod_{j=1}^m f_j(x) d\mu(x) \leq \prod_{j=1}^m ||f_j(x)||_{L^{p_j}} \qquad (2.2)$$

for all nonnegative functions $f_j \in L^{p_j}(\mathbb{R}^n)$ with $\sum_{j=1}^m p_j^{-1} = 1$.

The detail of the proof of (2.2) is given to Section 3. The second result concerning the monotonicity of $W = W(\tau, f)$-entropy is the following:

Theorem 2.2 *Let u be a positive solution of the heat equation (1.1) with* $\int_{\mathbb{R}^n} u \; d\mu = 1,\ \partial_t f = \Delta f - |\nabla f|^2 - \frac{n}{2\tau},\quad \tau > 0$ *and* $\frac{d\tau}{dt} = 1.$

Then $\frac{dW}{dt} \leq -\frac{2\tau}{n}\int_{\mathbb{R}^n} (|\nabla \ln\ u|^2 - \frac{n}{2\tau})^2 u \; d\mu.$ (2.3)

The monotonicity formula is sharp, indeed, equality holds in (2.3) if u should be taken to be the fundamental solution. This is verifiable by using $f(t,x) = \frac{|x|^2}{4t}$. The formula (2.3) is used here to derive a sharp logarithmic Sobolev inequality due to Gross [7]. The result is the following sharp inequality.

Theorem 2.3 *Let* $\phi \in L^2(\mathbb{R}^n, d\mu)$ *such that* $|\nabla\phi| \in L^2(\mathbb{R}^n, d\mu)$.

Then $\int_{\mathbb{R}^n} \phi^2 \ln|\phi| \; d\mu \leq \frac{\varepsilon}{2\pi}\int_{\mathbb{R}^n} |\nabla\phi|^2 \; d\mu + C(\varepsilon, n)$,(2.4)

where $\varepsilon > 0$ and $C(\varepsilon, n)$ is a constant depending on on ε and n.

3 MONOTONICITY OF $\Phi(t)$ AND THE PROOF OF THEOREM 2.1

The aim of this section is to prove the inequality in (2.2). Here, the fundamental solution of the heat equation is used. Let $f(t,x) = P_t f(x)$ solve the heat equation, where P_t is a one-parameter heat diffusion semigroup generated by Δ.

Setting $v := \log f(x)$ at $t = 0$, we have the initial value problem (from the heat equation)

$$\begin{cases} \frac{\partial v}{\partial t} = \Delta v + |\nabla v|^2 \\ v|_{t=0} = \log f \end{cases} \qquad (3.1)$$

with the diffusion semigroup $v(t,x) = \log P_t f(x)$. Following the idea first introduced in [6], a nonlinear heat semigroup can be defined by

$$f(t,x) = (P_t f(x)^2)^{\frac{1}{2}}$$

to obtain a nonlinear heat flow

$$\frac{\partial f(t,x)}{\partial t}|_{t=0} = \Delta f(x) + \frac{|\nabla f(x)|^2}{f(x)}.$$

Using the transformation of $\mathbb{R}^n$ onto j^{th} coordinates, $1 \leq j \leq m$, the j^{th} coordinate nonlinear heat flow is precisely written as

$$\frac{\partial f_j(t,x_j)}{\partial t} = \Delta f_j(t,x_j) + \frac{|\nabla f_j(t,x_j)|^2}{f_j(t,x_j)}. \qquad (3.2)$$

Now define the functional

$$\Phi(t) = \int_{\mathbb{R}^n} \prod_{j=1}^{m} f_j(t,x) \ d\mu(x) \tag{3.3}$$

which is known to be differentiable and smoothly continuous by the smoothing properties of the heat kernel semigroup for all $t > 0$.

3.1 Monotonicity formula for $\boldsymbol{\Phi(t)}$

Lemma 3.1 *Let $v_j\colon [0,\infty) \times \mathbb{R}^n \to \mathbb{R}, \quad 1 \le j \le m$ be a nonnegative solution of (3.1). Then $\Phi(t)$ is nondecreasing in time and specifically*

$$\Phi'(t) = \frac{1}{2}\int_{\mathbb{R}^n} \sum_{l\neq k} [\nabla_l v_k - \nabla_k v_l]^2 \prod_{j=1}^{m} f_j(t,x) \ d\mu(x). \tag{3.4}$$

Proof. Taking time derivative of $\Phi(t)$ and using (3.2), we have

$$\Phi'(t) = \frac{d}{dt}\left(\int_{\mathbb{R}^n}\prod_{j=1}^{m} f_j \ d\mu(x)\right) = \int_{\mathbb{R}^n}\left(\sum_{k=1}^{m}\frac{\partial}{\partial t}f_k\right)\prod_{j=1,j\neq k}^{m} f_j \ d\mu(x)$$

$$= \int_{\mathbb{R}^n}\sum_{k=1}^{m}(\Delta f_k + \frac{|\nabla f_k|^2}{f_k})\prod_{j=1,j\neq k}^{m} f_j \ d\mu(x)$$

$$= \int_{\mathbb{R}^n}\sum_{k=1}^{m}(\Delta f_k)\prod_{j=1,j\neq k}^{m} f_j \ d\mu(x) + \int_{\mathbb{R}^n}\sum_{k=1}^{m}(\frac{|\nabla f_k|^2}{f_k})\prod_{j=1,j\neq k}^{m} f_j \ d\mu(x).$$

Applying integration by parts on the first integral yields

$$\Phi'(t) = -\int_{\mathbb{R}^n}\sum_{k,l=1}^{m}(\nabla f_k, \nabla f_l)\prod_{j=1,j\neq k,l}^{m} f_j \ d\mu(x) + \int_{\mathbb{R}^n}\sum_{k=1}^{m}(\frac{|\nabla f_k|^2}{f_k})\prod_{j=1,j\neq k}^{m} f_j \ d\mu(x)$$

$$= -\int_{\mathbb{R}^n}\sum_{k,l=1}^{m}[\frac{\nabla f_k}{f_k}\cdot\frac{\nabla f_l}{f_l} - \frac{|\nabla f_k|^2}{f_k^2}]\prod_{j=1}^{m} f_j \ d\mu(x)$$

$$= \frac{1}{2}\int_{\mathbb{R}^n}\sum_{k,l=1,k\neq l}^{m}[\frac{|\nabla f_k|^2}{f_k^2} + \frac{|\nabla f_l|^2}{f_l^2} - 2\frac{\nabla f_k}{f_k}\cdot\frac{\nabla f_l}{f_l}]\prod_{j=1}^{m} f_j \ d\mu(x)$$

$$= \frac{1}{2}\int_{\mathbb{R}^n}\sum_{k\neq l}^{m}[\frac{\nabla f_k}{f_k} - \frac{\nabla f_l}{f_l}]^2\prod_{j=1}^{m} f_j \ d\mu(x) = \frac{1}{2}\int_{\mathbb{R}^n}\sum_{k\neq l}^{m}[\nabla v_k - \nabla v_l]^2\prod_{j=1}^{m} f_j \ d\mu(x).$$

There is equality in (3.4) if and only if

$$\frac{\nabla f_k}{f_k} - \frac{\nabla f_l}{f_l} = 0.$$

Since it is known that each f_j is strictly positive and each v_k is positive, smooth and bounded for all time $t > 0$, we therefore conclude that the quantity $\Phi(t)$ is nondecreasing for all $t > 0$.

Proof of Theorem 2.1

For any nonnegative measurable function f_j, $1 \leq j \leq m$, it seen that the functional $\Phi(t)$ in (3.3) is nondecreasing for all $t > 0$. Then, by the monotonicity property of $\Phi(t)$, the quantity

$$\widetilde{\Phi}(t) = \int_{\mathbb{R}^n} \prod_{j=1}^{m} f_j^{\frac{1}{p_j}}(t,x)d\mu(x)$$

is also nondecreasing. Indeed, taking time derivative of $\widetilde{\Phi}(t)$ as follows:

$$\widetilde{\Phi}'(t) = \int_{\mathbb{R}^n} \sum_{k=1}^{m} \frac{1}{p_k f_k}(\frac{\partial}{\partial t} f_k) \prod_{j=1}^{m} f_j^{\frac{1}{p_j}} d\mu.$$

Setting $v = \log f$ as before, where f solves the heat equation, then

$$f_j = e^{v_j} \ \forall j, \qquad \prod_{j=1}^{m} f_j^{\frac{1}{p_j}} = e^{\sum_{j=1}^{m} \frac{1}{p_j} v_j} \text{ and } \frac{\partial}{\partial t} f_k = \Delta f_k = \Delta v_k + |\nabla v_k|^2 f_k$$

Therefore

$$\widetilde{\Phi}'(t) = \int_{\mathbb{R}^n} [\sum_{k=1}^{m} \frac{1}{p_k}(\Delta v_k + |\nabla v_k|^2)] e^{\sum_{j=1}^{m} \frac{1}{p_j} v_j} d\mu$$

$$= \int_{\mathbb{R}^n} [-|\sum_{k=1}^{m} \frac{1}{p_k} \nabla v_k|^2 + \sum_{k=1}^{m} \frac{1}{p_k} |\nabla v_k|^2] \ e^{\sum_{j=1}^{m} \frac{1}{p_j} v_j}$$

by using integration by parts. Rewriting the first term on the right hand side of the last equation as follows (since $\max_k p_k^{-1} \leq 1$)

$$|\sum_{k=1}^{m} \frac{1}{p_k} \nabla v_k|^2 \leq \sum_{k=1}^{m} \frac{1}{p_k} |\nabla v_k|^2$$

reveals that $\widetilde{\Phi}'(t) \geq 0$. Therefore

$$\limsup_{t \to 0} \widetilde{\Phi}(t) \leq \liminf_{t \to \infty} \widetilde{\Phi}(t).$$

By Fatou's lemma, we have

$$\int_{\mathbb{R}^n} \prod_{j=1}^{m} f_j^{\frac{1}{p_j}}(x) \leq \limsup_{t \to 0^+} \widetilde{\Phi}(t). \qquad (3.5)$$

Indeed, there is equality in (3.5) since $\lim_{t\to 0} P_t f = f$. It then suffices to prove that

$$\liminf_{t\to\infty} \widetilde{\Phi}(t) \le \prod_{j=1}^{m} \left(\int_{\mathbb{R}^n} f_j\right)^{\frac{1}{p_j}}. \qquad (3.6)$$

The proof of (3.6) can be made more rigorous but outline is given here. Now, observe that $f_j(t,x)$ depends on x_j coordinate not on x itself, then we have

$$f_j(t,x_j) = (4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^{n_j}} e^{-|x_j - z|^2/4t} f_j(z)\ d\mu(z).$$

Notice that each f_j above solves the heat equation with initial condition $f_j(0,x) = f_j(x)$.

It is then written

$$\widetilde{\Phi}(t) = (4\pi t)^{-\frac{\sum_{j=1}^{m} \frac{n}{p_j}}{2}} \int_{\mathbb{R}^n} \prod_{j=1}^{m} \left(\int_{\mathbb{R}^n} e^{-||x_j - z||^2/4t} f_j(z)\ d\mu(z)\right)^{\frac{1}{p_j}}\ d\mu(x).$$

Noting also that $\sum_{j=1}^{m} \frac{n}{p_j} = n$. By rescaling argument, using $u_\epsilon(x) \longrightarrow \epsilon^{-n} v(\frac{x}{\epsilon})$, $\epsilon > 0$, (i.e., by making the change of variables $x = \epsilon y, \quad dx = \epsilon dy$), we then have the transformation

$$\widetilde{\Phi}(t) = \frac{\epsilon^n}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \prod_{j=1}^{m} \left(\int_{\mathbb{R}^n} e^{-\frac{\epsilon^2}{4t}||y_j - z/\epsilon||^2} f_j(z)\ d\mu(z)\right)^{\frac{1}{p_j}}\ d\mu(y).$$

Choosing a scaling factor $\epsilon^2 = 4\pi t$, by convolution property and Fubini's theorem yield

$$\liminf_{t\to\infty} \widetilde{\Phi}(t) \le \prod_{j=1}^{m} \left(\int_{\mathbb{R}^n} |f_j(z)|\ d\mu(z)\right)^{\frac{1}{p_j}} \int_{\mathbb{R}^n} \left(e^{-\pi||y||^2}\right)^{\frac{1}{p_j}} d\mu(y).$$

The claim (3.6) then follows immediately, since $\sum_{j=1}^{m} \frac{1}{p_j} = 1$ by the hypothesis of the theorem and $\int_{\mathbb{R}^n} e^{-\pi||y||^2} d\mu(y) = 1$ by standard Gauss integral.

4 PROOF OF MAIN RESULT ABOUT $W(\tau, f)$

This section is devoted to the proof of the monotonicity formula for $W(\tau, f)$. The relationship between the monotonicity formula and the classical entropy of Shannon H and Fisher information I are also highlighted.

4.1 Monotonicity of $W(\tau, f)$

Firstly, an important lemma that will be applied in the proof of Theorem 2.2 is stated and proved. The lemma also contains the proof of Proposition 1.1.

Lemma 4.1 *Let u be a Gaussian density function satisfying the heat equation (1.1). Then*

$$\frac{\partial}{\partial t}\left(\int_{\mathbb{R}^n} u\ln\, u d\mu\right) = -\int_{\mathbb{R}^n} |\nabla \ln\, u|^2 u\ d\mu \qquad (4.1)$$

and

$$\frac{\partial}{\partial t}\left(\int_{\mathbb{R}^n} |\nabla \ln\, u|^2 u\ d\mu\right) = -2\int_{\mathbb{R}^n} |\nabla\nabla \ln\, u|^2 u\ d\mu. \quad (4.2)$$

Proof of Lemma 4.1

By direct calculation and the following Bochner identity on $\mathbb{R}^n$

$$\frac{1}{2}\Delta|\nabla f|^2 = |\nabla\nabla f|^2 + \langle \nabla f, \nabla \Delta f\rangle$$

for every $f \in C^\infty$.

The proof of Theorem 2.2 is now stated.

Proof of Theorem 2.2

Using the heat equation we can rewrite the quantity (1.2) as

$$W = \int_{\mathbb{R}^n} \tau |\nabla \ln u|^2 u\ d\mu - \int_{\mathbb{R}^n} \ln(u)\ u\ d\mu - \frac{n}{2}\ln(4\pi\tau) - n$$
$$= \tau I(u) + H(u) - \frac{n}{2}\ln(4\pi\tau) - n.$$

Combining Lemma 4.1 and Propositon 1.1, the time derivative of W is obtained as follows:

$$\frac{dW}{dt} = \tau\frac{dI}{dt} + I + \frac{dH}{dt} - \frac{n}{2\tau}$$
$$= -2\tau J + 2I - \frac{n}{2\tau}$$
$$\leq -\frac{2}{n}\tau I^2 + 2I - \frac{n}{2\tau} = -\frac{2\tau}{n}\left(I - \frac{n}{2\tau}\right)^2.$$

Using $I=\int_{\mathbb{R}^n}|\nabla \ln u|^2 d\mu$, $\int_{\mathbb{R}^n} u d\mu = 1$ and Jensen's inequality, we have

$$(I-\frac{n}{2\tau})^2 = (\int_{\mathbb{R}^n}(|\nabla \ln u|^2-\frac{n}{2\tau})u)^2 \leq \int_{\mathbb{R}^n}(|\nabla \ln u|^2-\frac{n}{2\tau})^2 u \; d\mu.$$

This completes the proof of Theorem 2.2

4.2 Scaling for $\boldsymbol{W}$

Consider a standard mollifier $u(x)$ such that $\int_{\mathbb{R}^n} u(x)dx = 1$ and $u_\epsilon(x) = \epsilon^{-n}u(\frac{x}{\epsilon})$, $\epsilon > 0$. Then, Shannon entropy and Fisher information scale respectively as follows

$$H(u_\epsilon) = -\int_{\mathbb{R}^n} u_\epsilon \ln u_\epsilon d\mu = -\int_{\mathbb{R}^n}\epsilon^{-n}u(\frac{x}{\epsilon})\ln(\epsilon^{-n}u(\frac{x}{\epsilon}))d\mu$$

$$= -\int_{\mathbb{R}^n}\epsilon^{-n}u(\frac{x}{\epsilon})\ln u(\frac{x}{\epsilon}) - \int_{\mathbb{R}^n}\epsilon^{-n}u(\frac{x}{\epsilon})\ln\epsilon^{-n}d\mu$$

$$= -\int_{\mathbb{R}^n} u(y)\ln u(y)d\mu(y) - \ln\epsilon^{-n}\int_{\mathbb{R}^n} u(y)\; d\mu(y) = H(u) + n\ln\epsilon$$

and

$$I(u_\epsilon) = \int_{\mathbb{R}^n}\frac{|\nabla u_\epsilon|^2}{u_\epsilon}d\mu = \int_{\mathbb{R}^n}\frac{|\nabla \epsilon^{-n}u(\frac{x}{\epsilon})|^2}{\epsilon^{-n}u(\frac{x}{\epsilon})}d\mu$$

$$= \int_{\mathbb{R}^n}\frac{\epsilon^{-2(n+1)}|\nabla u(y)|^2}{\epsilon^{-n}u(y)}d\mu = \epsilon^{-2}\int_{\mathbb{R}^n}\frac{|\nabla u(y)|^2}{u(y)}d\mu = \epsilon^{-2}I(u).$$

It was already shown that entropy functional W relates well with Shannon entropy and Fisher information, but the scaling done above indicates that both $H(u)$ and $I(u)$ are not scale invariant with respect to dilation factor $u(u) \to u_\epsilon(x) = \epsilon^{-n}u(\frac{x}{\epsilon})$. Hence, the need to establish the scale invariance for W. Moreover, the decreasing in time monotonicity depends on scale invariance with respect to the dilation factor.

Recall

$$W = \int_{\mathbb{R}^n}(\tau\frac{|\nabla u|^2}{u^2} - \ln u)u \; d\mu - \frac{n}{2}\ln(4\pi\tau) - n.$$

Using the scale factor $u(x) \to u_\epsilon(x) = \epsilon^{-n}u(\frac{x}{\epsilon})$, we have

$$W_\epsilon = W(u_\epsilon,\tau) = \int_{\mathbb{R}^n}(\tau\frac{|\nabla u_\epsilon|^2}{u_\epsilon^2} - \ln u_\epsilon)u_\epsilon \; d\mu - \frac{n}{2}\ln(4\pi\tau) - n$$
$$= \tau I(u_\epsilon) + H(u_\epsilon) - \frac{n}{2}\ln(4\pi\tau) - n$$
$$= \tau\epsilon^{-2}I + H + n\ln\epsilon - \frac{n}{2}\ln(4\pi\tau) - n.$$

Choosing $\epsilon^2 = 4\pi\tau$, we have

$$W(u_\epsilon, \tau) = \int_{\mathbb{R}^n} [\frac{1}{4\pi}\frac{|\nabla v|^2}{v^2} - \ln v]v \ d\mu(y) - n,$$

where $u_\epsilon(x) = \epsilon^{-n}v(\frac{x}{\epsilon})$ with $dx = \epsilon^n dy$ and $v(y) = \epsilon^n u(\epsilon y)$ have been used in the last equation.

Now, suppose at time $t = 0, u_0(x) > 0$ and $\int_{\mathbb{R}^n} u_0(x)dx = 1$, then by the convolution of the heat kernel, we have

$$u(t,x) = (4\pi t)^{-\frac{n}{2}}\int_{\mathbb{R}^n} e^{-\frac{|x-z|^2}{4t}} u_0(z)dz$$

and

$$v(t,y) = \epsilon^n u(t,\epsilon y) = \epsilon^n(4\pi t)^{-\frac{n}{2}}\int_{\mathbb{R}^n} e^{-\frac{|\epsilon y-z|^2}{4t}} u_0(z)dz$$

$$= (4\pi t)^{\frac{n}{2}}(4\pi t)^{-\frac{n}{2}}\int_{\mathbb{R}^n} e^{-\frac{\epsilon^2}{4t}|y-z/\epsilon|^2} u_0(z)dz$$

$$= \int_{\mathbb{R}^n} e^{-\pi|y-z/\epsilon|^2} u_0(z)dz.$$

Since $z/\epsilon \to 0$ as $t \to \infty$, then $v(y) \to e^{-\pi|y|^2}$ pointwisely for fixed y and the W-entropy can be written as

$$W = \int_{\mathbb{R}^n} (\frac{1}{4\pi}\frac{|\nabla v(y)|^2}{v(y)^2} - \ln v(y))v(y) \ d\mu(y) - n.$$

It is also clear that $\nabla v(y) \to \nabla e^{-\pi|y|^2} = -2\pi y e^{-\pi|y|^2}$ and $\ln v(y) \to \ln e^{-\pi|y|^2} = -\pi|y|^2$ as $t \to \infty$. Therefore, by application of dominated convergence theorem as $t \to \infty$

$$W \to \int_{\mathbb{R}^n} 2\pi|y|^2 e^{-\pi|y|^2} d\mu(y) - n = 2\pi\int_{\mathbb{R}^n} |y|^2 e^{-\pi|y|^2} d\mu(y) - n = 0.$$

Combining the above with monotone decreasing property of W, we can conclude that

$$0 = \lim_{t\to\infty} W\tau(t), f(t) \leq W(\tau(t), f(t)).$$

This then implies that $W(f,\tau) \geq 0$ for all $t \in (0,\infty)$.

4.3 Log-Sobolev Inequality

Note that Log-Sobolev inequality (2.4) is scale invariant, that is, the inequality is preserved under multiplication by a positive constant. Once $W \geq 0$, one can then write:

$$\tau \int_{\mathbb{R}^n} \frac{|\nabla u|^2}{u} \; d\mu \geq \int_{\mathbb{R}^n} u \ln u \; d\mu + \frac{n}{2}\ln(4\pi\tau) + n \qquad (4.3)$$

which is essentially the Log-Sobolev inequality. Choosing a standard notation $\phi = \sqrt{u}$ with $\int_{\mathbb{R}^n} \phi^2 d\mu = 1$, it is then obtained at once

$$4\tau \int_{\mathbb{R}^n} |\nabla \phi|^2 d\mu \geq \int_{\mathbb{R}^n} \phi^2 \ln \phi^2 \; d\mu + \frac{n}{2}\ln(4\pi\tau) + n,$$

which implies

$$\int_{\mathbb{R}^n} \phi^2 \ln \phi^2 \; d\mu \leq \frac{\epsilon^2}{\pi} \int_{\mathbb{R}^n} |\nabla \phi|^2 d\mu + C(\epsilon, n). \qquad (4.4)$$

The above argument proves Theorem 2.3

5. CONCLUSION

This article constructed two entropy monotonicity formulas through the heat flows. The entropy formulas are employed to derive some functional inequalities. The first monotonicity formula constructed was combined with convolution and diffusion semigroup properties of the heat kernel to establish the proof of the multilinear Hölder inequalities. It was also shown that the second entropy monotonicity formula constructed is intimately related to the concavity of the power of Shannon entropy and Fisher Information, from which the associated logarithmic Sobolev inequality for probability measure is recovered. This work discovered that many functional and geometric inequalities can be retrieved as consequences of monotonicity properties of heat flow entropies. This approach is more simplified than existing methods like rearrangement or using either heat flow or entropy fomula independently.

REFERENCES

[1] A. Abolarinwa, N. Oladejo, S. Salawu, C. Onate, Logarithmic-Sobolev and multilinear Hölder's inequalities via heat flow monotonicity formulas, *App. Math. Comp.*, **364** (2020), 12460

[2] A. Abolarinwa , Differential Harnack and logarithmic Sobolev inequalities along Ricci- harmonic map flow, Pacific J. Math. 278(2) (2015), 257–290.

[3] A. Abolarinwa, N. Oladejo, S. Salawu, 2018. On the entropy formulas and solitons for the Ricci-harmonic flow, Bull. Irann. Math. Soc. In press.

[4] J. Bennet, Heat-flow monotonicity related to some inequalities in euclidean analysis, Contemporary Mathematics 505 85(96) (2010).

[5] J. Bennet, A. Carbery, M. Christ, T. Tao, Finite Bounds for H$\ddot{o}$lders-Brascamp-Lieb Multilinear Inequalities. preprint, math.MG/0505691.

[6] E.A. Carlen, E.H. Lieb, and M. Loss, A sharp analog of Young's inequality on S^N and related entropy inequalities. J. Geom. Anal. 14 (3) 487(520) (2004).

[7] L. Gross Logarithmic Sobolev inequalities, America J. Math 97(1)(1975),1061-1083.

[8] L. Gross Logarithmic Sobolev inequalities and contractivity properties of semigroups. Dirichlet forms (Varenna, 1992) Lecture Notes in Math. 1563.

[9] M. Ledoux, Heat flows, geometric and functional inequalities, Proceedings of the international congress of mathematicians. Vol. IV: Invited Lectures (Seoul 2014), Kyung Moon, Seoul, 117–135.

[10] L. H. Loomis and H. Whitney, An inequality related to the isoperimetric inequality, Bull. Amer. Math. Soc 55, (1949). 961-962.

[11] L. Ni, The Entropy Formula for Linear Heat Equation, Journal of Geom. Analysis 14(2)(2004), 85- 98.

[12] G. Perelman, The entropy formula for the Ricci Flow and its geometric application, arXiv:math.DG/0211159v1 (2002).

[13] C. E. Shannon, A mathematical theory of communication. The Bell Syst. Tech. J. 27,(1948), 379-423.

[14] A.J. Stam, Some inequalities satisfied by the quantities of information of Fisher and Shannon. Inf. Contr. 2, (1959) 101-112.

[15] G. Toscani, Lyapunov functionals for the heat equation and sharp inequalities, Atti della Accademia Peloritana dei Pericolanti, 91(1), (2013)

[16] G. Toscani, An information-theoretic proof of Nash's inequality, Rend. Lincei Mat. Appl., 24 (2013), 83–93.

[17] C. Villani, A short proof of the concavity of entropy power. IEEE Trans. Info. Theory 46, (2000)(4),1695-1696,.