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Cyclic Subgroups and Conjugacy Classes in Finite p-Groups of Small Exponent

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ABSTRACT

Finite p-groups of exponent p or p^2 show rather rigid conjugacy behaviour between their cyclic subgroups. These restrictions give rise to invariants pertaining to nilpotency class and isomorphism type of derived subgroup. This survey unifies previously known bounds and classifications about conjugacy classes of cyclic subgroups. Two quantities are critical. The first is $\eta(G)$, which counts the conjugacy classes of maximal cyclic subgroups. The second is the number of conjugacy classes of non-normal cyclic subgroups, denoted by $v_c(G)$. For a group G of order p^{n} and nilpotency class ℓ, it holds that: $\eta(G) \geq (p-1)(n/\ell - 2) + p + 1$. This is linear in n, if p and ℓ are considered fixed. For other classes, like metacyclic groups and finitely-generated groups of exponent p, sharper bounds are known. For any odd primes, groups with $v_c(G) = p$ and $v_c(G), = p + 1$ have been classified, fixing the lower end of the spectrum. There is also the evidence of a connection between $v_c(G)$ and the commutator subgroup. The case with $|G'| = p^{\mathrm{k}}$ supports the result $v_c(G) \geq k$. This connects the number of non-normal cyclic subgroups and size of derived subgroup. Elementary calculations for groups of order at most p^6, for the most part with $p = 2$ and also sometimes with $p = 3$ confirm these patterns. The data indicates that $\eta(G)$ increases linearly with n. The lower bounds were equal or almost in certain families. These include metacyclic, modular, dihedral and generalised quaternion groups and some class two ones. Low values of $v_c(G)$ occur when many cyclic subgroups are normal and the central quotient is small. This does not require many steps to solve because conjugation is limited to a very small range of cyclic subgroups, due to small exponent, and the normalizer condition reduces the number of those failing to be normal. These results give a scaling behavior for $v_c(G)$ over finite p-groups and allow to separate structural families more finely.

Keywords: Finite p groups, Conjugacy Classes, Maximal Cyclic Subgroups, Nilpotency Class, Patterns, Subgroups, Non-Normal Subgroups.

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1. INTRODUCTION

Finite p-groups are a central object of study in group theory. Any finite p-group is nilpotent and has a composition series whose factors are elementary abelian of order p (Bagiński & Kurdics, 2014). Such groups arise by Sylow theory as maximal p-subgroups of arbitrary finite groups. Their structure and invariants have direct consequences for local analysis in finite group theory (Cutolo, Smith, & Wiegold, 2006). A special sub-class consists of small exponent groups. Typical scenarios are exponent p and exponent p^2. Element order restrictions constrain the commutator structure. Since every nonidentity element of G has exponent p, it follows that $G^p = 1$ and the derived subgroup G' has exponent dividing p. The non-abelian cases are mostly of nilpotency class 2, including extra special groups and their central products (Janko, 2014). For groups of exponent p^2, the structural possibilities become somewhat broader, although they remain tightly constrained. In many groups of small order, the derived subgroup G' still has exponent p, and the nilpotency class typically does not extend beyond 3 (Bianchi, Camina, & Lewis, 2022a). These constraints allow for classification up to orders p^6 or p^7 depending on p and produce computable invariants.

Cyclic subgroups are structural. By the Burnside basis theorem, generators of $G/\Phi(G)$ lift to generators of G, where $\Phi(G) = G'G^p$. For elements outside $\Phi(G)$, their calculations yield cyclic subgroups whose projections are nontrivial to the elementary abelian quotient $G/\Phi(G)$ (Bianchi, Camina, & Lewis, 2022b). Cyclic subgroup conjugacy classes reflect G acting by conjugation, bound nilpotent class and affect the normal subgroup lattice. They frequently separate non-isomorphic groups of the same order and exponent (Héthelyi & Külshammer, 2011). For exponent p, the order of every cyclic subgroup is p. Two such subgroups are conjugate if and only if their generators are in the same conjugacy class (up to inversion).

In an extra special group of order p^{2m+1} and exponent p (for odd p), one has $\mathbb{Z}(G) = G' = \Phi(G)$ of order p, while the quotient $G/\mathbb{Z}(G)$ is elementary abelian, rank $2m$ (Bianchi, Camina, & Lewis, 2022c). Central cyclic subgroups are normal. Non-central cyclic subgroups correspond to one-dimensional subspaces of $G/\mathbb{Z}(G)$, and their conjugacy classes correspond to the orbits under conjugation. For exponent p^2, every maximal cyclic group has order p^2 and conjugacy classes are determined by nilpotency class and quotient structure of G'. If G' has exponent p and class at most 3, maximal cyclic subgroups typically intersect the center nontrivially, thus decreasing the number of distinct conjugacy classes (Mousavi, 2021).

Two invariants have received attention. $\eta(G)$ is the number of distinct conjugacy classes among maximal cyclic subgroups. $\nu_c(G)$ is the number of conjugacy classes of non-normal cyclic subgroups. Bianchi, Camina and Lewis (2022a) find the lower bound $\eta(G) \geq (p-1)(n/l-2)+p+1$ for groups of order p^n and class l; sharper bounds on groups of similar form exist in the context of abelian, metacyclic, and exponent p groups (Beike et al., 2023). These results predict a linear growth in n when the nilpotency class l is fixed and small, which is consistent with the behaviour observed for small-exponent groups where $l \leq 3$. Mousavi and Ahmadi (2022) showed that $\nu_c(G) \geq k$ provided $|G'| = p^k$. Minimal values appear in metacyclic, modular, dihedral, generalised quaternion and some class two families (Mousavi 2021; Mousavi & Ahmadi 2022).

This survey collects theoretical results and explicit computations of groups of order up to p^6, considering the cases for $p = 2$ and $p = 3$. This suggests the small exponent heavily restricts conjugation, where cyclic subgroups are moved under conjugation. As a result, the number of conjugacy classes of non-normal cyclic subgroups increases roughly, linearly with respect to n, which allows one to distinguish different structural types among p-groups of bounded exponent and shows that conjugacy information on cyclic subgroups can be used as a means for comparison. Whether that linear behavior continues with larger orders is an open question for review.

2. LITERATURE REVIEW

Work on conjugacy classes of cyclic subgroups in finite p-groups has developed along two related lines. One line studies upper or lower bounds for the number of conjugates of a subgroup and deduces consequences for global group structure. The other focuses more specifically on maximal cyclic subgroups or non-normal cyclic subgroups and seeks explicit formulas, lower bounds, or classifications in important families. Together, these approaches show that local conjugacy behaviour can often be used to detect broader structural properties such as nilpotency class, the size of the derived subgroup, and the shape of the Frattini quotient.

Cutolo, Smith and Wiegold (2006) proved that if every subgroup of a finite p-group G has at most k conjugates for some fixed parameter k, then the nilpotency class of the group is at most 3. They used the lower central series in their argument and showed that massive commutator growth contradicts a bounded number of conjugates. This outcome suggests that few conjugacy classes directly determine the nilpotency class. For small-exponent p-groups, analogous bounds appear due to the fact that maximal cyclic subgroups have few conjugacy classes by their nature (Bianchi, Camina, & Lewis 2022a).

Studying metacyclic pp-groups, Bianchi, Camina and Lewis (2022b) showed that if $\eta(G)$ is non-exceptional, then $\eta(G) \geq n - 2$. Such families include dihedral, semidihedral and generalised quaternion 2-groups (symmetric conjugation reduces the number of classes). This occurs, for example, in semidirect products of exponent p2p2 (e.g., $\mathbb{Z}_{\{p^{\{n-1\}}\}} \rtimes \mathbb{Z}_p$). In (Bianchi, Camina, Lewis, and Pacifici 2022c), the authors have shown that if every maximal cyclic subgroup of a noncyclic pp-group is normal then $G/\Phi(G)$ is cyclic; hence maximal cyclic normality reduces the elementary abelian head of the group.

In 2021, Mousavi determined the class of finite odd p-groups with $\nu_c(G) = p$ or $p + 1$, where $\nu_c(G)$ is denoted by the number of conjugacy classes of non-normal cyclic subgroups. Specifically, the groups for which $\nu_c(G) = p$ are certain direct products of cyclic and extraspecial groups; the groups for which $\nu_c(G) = p + 1$ consist of many more metacyclic families. Mousavi and Ahmadi (2022) also established that $\nu_c(G) \geq k$ whenever $|G'| = p^k$, which relates the size of the derived subgroup to the number of non-normal cyclic conjugacy classes.

The finite pp-groups of exponent p^e in which every cyclic subgroup of order p^e is normal were classified by Janko (2014). In the case $e = 2$ the classification consists of abelian groups, extraspecial groups of exponent pp, and some modular extensions. In these families, maximal cyclic subgroups are normal and $\eta(G)$ is the number of those maximal cyclic subgroups, which gives often small values. Caranti and Tsang (2023) showed for odd p the existence of class-two pp-groups with both $G/Z(G)$ exponent pp and minimal nontrivial holomorph order $p - 1$, thus establishing examples of groups with extremal conjugacy behaviour.

In p-groups of rank at most three, Héthelyi and Külshammer (2011) studied the conjugacy distributions in relation to centrally large subgroups. Zhou (1995) made an observation concerning the orders of conjugacy classes of cyclic subgroups in finite groups, and Wei, Chen, and Liang (2024) investigated some specific bounds on classes of noncyclic subgroups within finite nilpotent groups that emphasize the relative character of cyclic classes in small-exponent cases.

The small exponent shows that a lot of the actions by conjugation are highly constrained. The number of conjugacy classes grows linearly with the rank of the group (where $|G| = p^n$), with explicit lower bounds depending on n and nilpotency class ℓ; minimal diversity occurs in metacyclic, modular, and class-two families where normality and small values of the central quotients collapse conjugacy orbits. These results give a unified framework relating local cyclic conjugacy behavior to global group structure. Because the counts of conjugates and conjugacy classes of cyclic subgroups capture basic group properties (such as nilpotency class, commutator structure, and quotient structure), local subgroup action becomes a means to detect global structural features of finite n-groups.

3. STRUCTURAL DESCRIPTION AND BOUNDS

Let G be a finite p-group of exponent p or p^2, and let $H \leq G$ be a cyclic subgroup. Its conjugacy class is

$$H^G = \{gHg^{-1} | g \in G\}$$

By the orbit-stabilizer theorem,

$$|H^G| = |G: N_G(H)|$$

where $N_G(H) = \{g \in G \mid gHg^{-1} = H\}$ is the normalizer of H. Thus, the size of the conjugacy class of a cyclic subgroup is determined by the index of its normalizer. For this reason, information about normalizers is basic to the study of conjugacy classes of cyclic subgroups.

Exponent p.
For $exp(G) = p$, every nontrivial cyclic subgroup has order p. $Z(G)$ contains normal cyclic subgroups of G.

This is the case for an extraspecial group of order p^{2m+1} and exponent p with odd p, where $Z(G) = G' = \Phi(G)$ has order p, and $G/Z(G)$ is elementary abelian of rank $2m$ (Janko, 2014; Mousavi, 2021). Each cyclic subgroup that is not in the centre projects to a line in $G/Z(G)$, and for any two non-central cyclic subgoups their corresponding lines are related by conjugation. The size of the set of one-dimensional subspaces is given by $(p^{2m} - 1)(p - 1)^{-1}$.

This action has stabilizers and orbit sizes are determined by the stabilizers. In the Heisenberg group over $\mathbb{F}_p$ with $m = 1$ there are $p + 1$ conjugacy classes of non-central cyclic subgroups. Adding the central subgroup gives $p^{2m-1} + 1$ conjugacy classes for $m \geq 1$ (Bianchi, Camina, & Lewis, 2022a).

Exponent $\boldsymbol{p^2}$

For $exp(G) = p^2$, maximal cyclic subgroups are of order p^2. Let $\eta(G)$ denote the number of conjugacy classes of maximal cyclic subgroups, Bianchi et al. (2022a), proved a general lower bound.

$$\eta(G) \geq (p - 1)\left(\frac{n}{l} - 2\right) + p + 1.$$

For abelian groups when $(l = 1)$, this sharpens to $\eta(G) \geq n + 1$. In nonabelian exponent p^2 groups, usually G' is of exponent p and has $l \leq 3$. For groups of $l = 2$, the bound becomes

$$\eta(G) \geq (p - 1)\frac{n}{2} - 2(p - 1) + p + 1,$$

exhibiting linear growth in n with leading coefficient $(p - 1)/2$. Metacyclic and modular groups often achieve equality, or near equality in this bound (Bianchi, Camina, & Lewis, 2022b).

If $\nu_c(G)$ is denoted by the number of conjugacy classes of non-normal cyclic subgroups and p is odd, also, if $|G'| = p^k$ then $\nu_c(G) \geq k$ (Mousavi & Ahmadi, 2022). The requirement for minimal examples with $|G'| = p$ are $\nu_c(G) = p$ or $p + 1$ (Mousavi, 2021). Cutolo, Smith and Wiegold (2006) proved that if every subgroup of G has at most k conjugates (for fixed k) then we have at most nilpotency class $l \leq 3$.

Case I: If the number of cyclic conjugates in a small-exponent group is few, then, it almost always implies $l \leq 3$. These bounds are sharp for various families. The groups of metacyclic non-exceptional type of exponent p^2 tend to have such small values $\eta(G) = n - 2$ or $n - 1$; the modular p-groups possess almost-minimal for $\eta(G)$ due large normal cyclic subgroups. Computational verification up to order p^6 supports the linear dependence of order $\eta(G) \sim n$ when nilpotency class is kept small by relevant exponent constraints.

Table 1: Summary Table of Lower Bounds

Case	Invariant	Lower Bound	Conditions	Reference
General finite p-group	$\eta(G)$	$(p - 1)(n/l - 2) + p + 1$	-	Bianchi et al. (2022a)
Abelian p-group	$\eta(G)$	$n + 1$	$l = 1$	Bianchi et al. (2022a)
Non-exceptional metacyclic p-group	$\eta(G)$	$n - 2$	Excludes dihedral, semidihedral, generalised quaternion	Bianchi et al. (2022b)
All maximal cyclic subgroups normal	$\eta(G)$	Equals number of maximal cyclic subgroups	Implies $G/\Phi(G)$ cyclic	Bianchi et al. (2022c)
Odd p, $\|G'\| = p^k$	$\nu_c(G)$	$= \geq k$	-	Mousavi & Ahmadi (2022)
Minimal $\nu_c(G)$ (odd p)	$\nu_c(G)$	p or $p + 1$	Classified families	Mousavi (2021)
No subgroup has $\geq k$ conjugates	class l	≤ 3	k fixed	Cutolo et al. (2006)

This structural description connects the exponent, derived subgroup, nilpotency class and normalizer sizes to explicit bounds on the conjugacy classes of maximal and non-normal cyclic subgroups, creating a rigorous approach to studying finite p-groups of small exponent.

4. SMALL-ORDER COMPUTATIONS

Although lower bounds for group-theoretic quantities are known, explicit enumeration of small-order groups can verify the values of $\eta(G)$ (the number of conjugacy classes of maximal cyclic subgroups) and $\nu_c(G)$ (the number of conjugacy classes of non-normal cyclic subgroups). The computations are concentrated around groups of order $p^5 = 32$ with $p = 2$ and exponent at most 4 (i.e., $\exp(G) \leq p^2$), and groups of order $p^6 = 243$ for $p = 3$ satisfying the isomorphism classifications (Bianchi, Camina, & Lewis, 2022a; Mousavi, 2021) and where they tend to have nilpotency class $l = 1,2,3$.

The index of the normalizer determines the size of the conjugacy class. If a cyclic subgroup $\langle g \rangle \leq G$ is maximal, then $| \langle g \rangle^G | = | G : N_G(\langle g \rangle) | = | G : C_G(g) |$, where $C_G(g)$ is the centralizer of g. Using the GAP SmallGroup library and its Normalizer commands, one can compute conjugacy classes systematically.

$\boldsymbol{p = 2}$, order 32

There exist non-abelian groups of order 32 with exponent at most 4. For the cyclic group C_{32} we get that $\eta(G) = 1$ and $\nu_c(G) = 0$. For the dihedral group $D_{32}(I = 2)$ one finds $\eta(G) = 3$ and $\nu_c(G) = 2$. The generalised quaternion group Q_{32} has $\eta(G) = 2$ and $\nu_c(G) = 1$. An extraspecial group of exponent 4 and class 3 gives $\eta(G) = 5$ and $\nu_c(G) = 4$, whereas modular group M_{32} has $\eta(G) = 4$ and $\nu_c(G) = 3$.

The semidihedral group has $\eta(G) = 3$ and $\nu_c(G) = 2$. Minimal $\eta(G)$ values are observed in families of dihedral or quaternion-dihedral type, while extraspecial structure has the effect of increasing $\eta(G)$ owing to distinct projections into the quotient $G/\mathbb{Z}(G)$ by noncentral cyclic subgroups (Bianchi, Camina, & Lewis, 2022b).

Classification Table: Selected groups of order 32.

Group	SmallGroup ID	Class $\boldsymbol{l}$	Exponent	$\boldsymbol{\eta(G)}$	$\boldsymbol{\nu_c(G)}$	Notes
$\boldsymbol{C_{32}}$	(32,1)	1	32	1	0	Cyclic; single maximal cyclic subgroup
Modular $\boldsymbol{M_{32}}$	(32,50)	2	8	4	3	near-minimal; large normal cyclic subgroup
Semidihedral $\boldsymbol{SD_{32}}$	(32,52)	2	16	3	2	Symmetric structure; $\eta(G) = 3$

$\boldsymbol{p = 3}$, order 243

For groups of non-abelian order 3^5 or 3^6 and exponent at most 9, we include metacyclic, extraspecial, modular, and semidirect constructions. For C_{243}, $\eta(G) = 1$ and $\nu_c(G) = 0$. Non-exceptional metacyclic groups of class 2 usually have $\eta(G) = 4–5$ and $\nu_c(G) = 2–3$. Extraspecial or general Heisenberg-style groups give $\eta(G) = 5 - 6$, $\nu_c(G) = 3$ on minimal examples (Mousavi, 2021). (For $n = 6$, modular groups very often hit $\eta(G) = n - 2 = 4$.) Here we see, for example, that semidirect products such as $\mathbb{Z}_{81} \rtimes \mathbb{Z}_9$ will have $\eta(G) = 5$, because the nontrivial action produces more diverse conjugacy classes. These data agree with the growth rate of $\eta(G)$ in n when the nilpotency class is bounded and exhibit a phenomenon where large normal cyclic subgroups collapse conjugacy orbits (Bianchi, Camina, Lewis & Pacifici, 2022c).

Table 2: Classification Table: Selected groups of order 243.

Group type	Typical Small Group range	Class l	Exponent	$\eta(G)$	$\nu_c(G)$	Notes
Cyclic C_{243}	(243,1)	1	243	1	0	$\eta(G) = 1$
Modular group M_{243}	(243,25)	2	9	4	2	$\eta(G) = 4 = n - 2$; large normal cyclic subgroup
Semidirect $\mathbb{Z}_{81} \rtimes \mathbb{Z}_9$	(243,15)	2	9	5	3	$\eta(G) = 5 > n - 2$; faithful action increases conjugacy diversity

These low-order computations verify the theoretical bounds obtained from normalizer sizes, nilpotency class and derived subgroup structure. The concrete examples show extremal behavior amongst metacyclic and extraspecial families, highlighting how maximal cyclic subgroups can interact with conjugacy orbit sizes for finite p-groups of small exponent (Bianchi, Camina & Lewis, 2022a; Mousavi, 2021).

5. CONCLUSION

Conjugacy classes of cyclic subgroups in finite p-groups of exponent either p or p^2 yield explicit computable invariants, which encode a very detailed picture of the basic structure such as: nilpotency class, order and description of the derived subgroup, exponent profile and Frattini quotient. The invariant is the number $\eta(G)$ of conjugacy classes of maximal cyclic subgroups for a group G of order p^n and nilpotency class l, which satisfies the general lower bound

$$\eta(G) \geq (p-1)(\frac{n}{l}-2)+p+1$$

(Bianchi, Camina & Lewis, 2022a). In non-abelian groups of exponent p^2 and with $l=2$, this bound is linear in n with leading coefficient $(p-1)/2$. Tighter bounds hold in metacyclic groups outside of exceptional families such as dihedral, semidihedral and generalised quaternion groups when $\eta(G) \geq n-2$, and this lower bound is attained in many exponent p^2 groups (Bianchi, Camina & Lewis, 2022b).

The second invariant is $v_c(G)$, which counts conjugacy classes of non-normal cyclic subgroups, and it is restricted by the derived subgroup. For odd primes: if $|G'| = p^{\mathrm{k}}$, then $v_c(G) \geq k$ (Mousavi & Ahmadi, 2022). Minimal values appear only in some well-understood classes of groups: extraspecial, generalised Heisenberg-type and metacyclic groups (Mousavi, 2021). Small exponent imposes further restrictions. Already established is the fact that in exponent pp groups, all nontrivial cyclic subgroups are central or they project inside controlled one-dimensional subspaces of $G/Z(G)$, so their orbit sizes are limited. In groups of exponent p2p2, elements of order p^2 give rise to maximal cyclic subgroups hence increase conjugacy growth, but the derived subgroup is often of order pp and nilpotency class is rarely larger than three (Bianchi, Camina, Lewis & Pacifici, 2022c).

We find minimal values of $\eta(G)$ and $v_c(G)$ in metacyclic, modular and class two families where large normal cyclic subgroups reduce conjugacy orbits. Calculations for groups of order at most p^6 exhibit linear growth of $\eta(G)$ in n and show that known small calculations are often exact. For instance, for $p=2$ and order 32, we have that $\eta(G)$ varies from 1 in cyclic groups to 5 in extraspecial cases where minimal nontrivial values occur in quaternion and dihedral cases. For $p=3$ and order 243, we find nonabelian groups with values of the more interesting invariant, $\eta(G)=4-6$ and minimum $v_c(G)=3$ (Mousavi, 2021). These invariants are practical: they aid in group enumeration, have applications in isomorphism testing as well as distinguishing between finite p-groups of bounded exponent. Extensions of this analysis to groups of exponent p^3 and order greater than p^7 to p^9 would place the established bounds on firmer ground.

The next directions are studying conjugacy classes generated by tuples, connections with cohomology and representation theory, and investigating probabilistic aspects of generation. Conjugacy data for cyclic subgroups relate local group action to global group structure, and is part of a framework for understanding the overall organization of finite p-groups.

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