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The First Eigenvalue of The Weighted p-Laplacian Under the Ricci-Harmonic Flow

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ABSTRACT

This paper examines the behaviour of the spectrum of the weighted p-Laplacian on a complete Riemannian manifold evolving by the Ricci-harmonic flow. Precisely, the first eigenvalue diverges in a finite time along this flow.

keywords:- Ricci harmonic flow, Laplace-Beltrami operator, eigenvalue, monotonicity, Ricci solitons.

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1. INTRODUCTION

In this paper we study maximal time behaviour of the first eigenvalue of the weighted p-Laplaian on an n-dimensional complete Riemannian manifold $(M, g, d\mu)$ equipped with weighted measure $d\mu = e^{-\phi} dv,\ \phi \in C^{\infty}(M, d\mu)$, whose metric $g = g(t)$ evolves along the Ricci-harmonic flow. It is a well known feature, that spectrum as an invariant quantity, evolves as the domain does under any geometric flow. It is found out that the bottom of the spectrum diverges in a finite time of the flow existence.

1.1 The Ricci-harmonic flow

The pair $(g = g(x,t), \phi = \phi(x,t))$ is said to Ricci-harmonic flow if it satisfies the system of quasilinear parabolic equations

$$\begin{cases} \frac{\partial}{\partial t} g(x,t) = -2Rc(x,t) + 2\alpha \nabla\phi(x,t) \otimes \nabla\phi(x,t) \\ \frac{\partial}{\partial t} \phi(x,t) = \Delta_g \phi(x,t), \end{cases} \tag{1.1}$$

subject to initial condition $(g(x,0), \phi(x,0)) = (g_0, \phi_0)$.

Here $\phi(x,t): M \times [0,\infty) \to \mathbb{R}$ is a one parameter family of smooth functions at least C^2 in x and C^1 in t, $\otimes$ is the tensor product, Rc is the Ricci curvature tensor of (M, g), ∇ is the gradient operator, α is a nonincreasing constant function of time bounded below by $\alpha_n > 0$ in time and Δ is the Laplace-Beltrami operator on M. The system (1.1) was first studied by List [12] with motivation coming from general relativity. It was generalized by Müller [13] to the situation where $\phi: (M, g) \to (N, h)$, ((N, h) is a compact Remannian manifold endowed with static metric h) and ϕ satisfies Eells and Sampson's heat flow [7] for harmonic map. Indeed, if ϕ is a constant function, the flow degenerates to the well known Hamilton's Ricci flow [9]. For detail discussion on Hamilton's Ricci flow see [4, 5] and for Eells and Sampson's heat flow see [8].

Recently, obtaining information about behaviours of eigenvalues of geometric operators on evolving manifolds has become a topic of concerns among geometers since these information usually turn to be useful in the study of geometry and topology of the underlying manifolds. Perelman [14] recorded a great success by proving that the first nonzero eigenvalue of $-4\Delta + R$ is nondecreasing along the Ricci flow via the monotonicity formula for his energy functional. Cao [3] extended Perelman's result to the first eigenvalue of $\Delta + \frac{R}{2}$ on the condition that the curvature operator is nonnegative. Later, Li [10] proved the same result without any curvature assumption. Recently, the first author studied the evolution and monotonicity of the first eigenvalue of p-Laplacian and weighted Laplacian in [1] and [2], respectively. He found some monotonic quantities under the respective flows. In [6] Cerbio and Fabricio proved that the first eigenvalue of Laplace-Beltrami operator on a 3-dimensional closed manifold with positive Ricci curvature diverges as $t \to T$ under the Hamilton's Ricci flow. The authors in [17] obtained a similar result under 3-dimensional Ricci-Bourguignon flow. In [2] the first author proved the same result for the weighted Laplacian under the Ricci-harmonic flow. Motivated by [6] and [2], we will show the same result for the eigenvalue of weighted p-Laplacian under the Ricci-harmonic flow and on gradient almost Ricci-harmonic soliton for $p = 2$.

2. PRELIMINARIES

2.1 Notation

Throughout, (M, g) will be taken to be a compact Riemannian manifold. Sometimes, our calculations are performed in a local coordinate system $\{x^1, x^2, \ldots, x^n\}$ with repeated indices summed up. The Riemannian metric $g(x)$ at any point $x \in M$ is a bilinear symmetric positive definite matrix written in local coordinates as $g(x) = g_{ij}\, dx^i dx^j$, where g_{ij} and $g^{ij} = (g_{ij})^{-1}$ are the components of the matric and its inverse, respectively.

We denote a symmetric 2-tensor by $Sc := Rc - \alpha \nabla\phi \otimes \nabla\phi$, its components by $S_{ij} := R_{ij} - \alpha\phi_i\phi_j$ and its metric trace by $S := R - \alpha|\nabla\phi|^2$, where R_{ij} are the Ricci tensor's components, R is the scalar curvature of (M, g) and $\phi_i = \nabla_i\phi = \frac{\partial}{\partial x^i}\phi$. We denote the Laplace-Beltrami operator, p-Laplacian, weighted Laplacian (ϕ-Laplacian) and weighted p-Laplacian on (M, g) by Δ, Δ_p, Δ_ϕ and $\Delta_{p,\phi}$, respectively. We denote dv as the Riemannian volume measure on (M, g) and $d\mu := e^{-\varphi(x)}\, dv$, the weighted volume measure, where $\varphi \in C^\infty(M)$.

2.2 Laplacian-type operator

Let $f : M \to \mathbb{R}$ be a smooth function, for $p \in [1, +\infty)$, the p-Laplacian of f is defined as

$$\begin{aligned}\Delta_p f &= \operatorname{div}(|\nabla f|^{p-2}\nabla f)\\ &= |\nabla f|^{p-2}\Delta f + (p-2)|\nabla f|^{p-4}\mathrm{Hess}f(\nabla f, \nabla f),\end{aligned}$$

where div is the divergence operator, the adjoint of gradient for the L^2-norm induced by the metric on the space of differential forms. When $p = 2$, Δ_p is the usual Laplace-Beltrami operator. For the weighted volume measure $d\mu = e^{-\phi}\, dv$, the ϕ-Laplacian is defined by

$$\Delta_\phi f := e^{\phi}\operatorname{div}(e^{-\phi}\nabla f) = \Delta f - \langle\nabla\phi, \nabla f\rangle,$$

which is a symmetric diffusion operator on $L^2(M, g, d\mu)$ and self-adjoint with respect to the measure in the sense of integration by parts formula

$$\int_M \Delta_\phi u v d\mu = -\int_M \langle \nabla u, \nabla v \rangle d\mu = \int_M u \Delta_\phi v d\mu$$

for any $u, v \in C^\infty(M)$. When ϕ is constant, the weighted Laplacian is just the the Laplace-Beltrami operator. The weighted p-Laplacian on smooth functions generalizes the p-Laplacian and the weighted Laplacian. It is defined by

$$\Delta_{p,\phi} := e^\phi \mathrm{div}(e^{-\phi} |\nabla f|^{p-2} \nabla f) = \Delta_p f - |\nabla f|^{p-2} \langle \nabla \phi, \nabla f \rangle.$$

When $p = 2$, this is just the weighted Laplacian and when ϕ is a constant, it is just the p-Laplacian.

The mini-max principle also holds for the weighted p-Laplacian where its first nonzero eigenvalue is charaterizeed as follows

$$\lambda_1(t) = \inf_f \{\int_M |\nabla f|^p d\mu : \int_M |f|^p d\mu = 1, f \neq 0, f \in W^{i,p}(M, g, d\mu)\} \quad (2.1)$$

satisfying the constraints $\int_M |f|^{p-2} f d\mu = 0$, where $W^{1,p}(M, g, d\mu)$ is the completion of

$C^\infty(M, g, d\mu)$ with respect to the norm

$$\| f \|_{W^{1,p}} = (\int_M |f|^p d\mu + \int_M |\nabla f|^p d\mu)^{\frac{1}{p}}.$$

The infimum in (2.1) is achieved by $f \in W^{1,p}$ satisfying the Euler-Lagrange equation

$$\Delta_{p,\phi} f = -\lambda_1 |f|^{p-2} f \quad (2.2)$$

or equivalently,

$$\int_M |\nabla f|^{p-2} \langle \nabla f, \nabla \psi \rangle d\mu - \lambda_1 \int_M |f|^{p-2} \langle f, \psi \rangle d\mu = 0 \quad (2.3)$$

for all $\psi \in C_0^\infty(M)$ in the sense of distribution. In other words, we say that λ is an eigenvalue of $\Delta_{p,\phi}$ and $f \in W^{1,p}$ is the corresponding eigenfunction if the pair (λ, f) satisfies (2.2). Then (2.3) implies

$$\int_M |\nabla f|^p d\mu = \lambda \int_M |f|^p d\mu \quad (2.4)$$

implying $\lambda = \int_M |\nabla f|^p d\mu$ since $\int_M |f|^p d\mu = 1$.

2.3 Regularization procedure

Firstly, we introduce the linearized operator of the weighted p-Laplacian on function $h \in C^\infty(M)$ defined pointwise at the points $\nabla h \neq 0$ [16]

$$\begin{aligned} \mathcal{L}_\phi(\tilde{f}) &:= e^\phi \mathrm{div}(e^{-\phi} |\nabla h|^{p-2} G(\nabla \tilde{f})) \\ &= |\nabla h|^{p-2} \Delta_\phi \tilde{f} + (p-2) |\nabla h|^{p-2} \mathrm{Hess} \tilde{f}(\nabla h, \nabla h) + (p-2) \Delta_{p,\phi} h \frac{\langle \nabla h, \nabla \tilde{f} \rangle}{|\nabla h|^2} \\ &+ 2(p-2) |\nabla h|^{p-4} \mathrm{Hess} \tilde{f}(\nabla h, \nabla \tilde{f} - \frac{\nabla h}{|\nabla h|} \langle \frac{\nabla h}{|\nabla h|}, \nabla \tilde{f} \rangle) \end{aligned}$$

for a smooth function $\tilde{f}$ on M, where G can be viewed as a tensor defined as

$$G := \mathrm{Id} + (p-2) \frac{\nabla h \otimes \nabla h}{|\nabla h|^2}.$$

Notice that $\mathcal{L}_\phi$ is positive definite for $p > 1$ and strictly elliptic in general at these points ($\nabla h \neq 0$), and that the sum of its second order part is

$$\mathbb{L}_\phi \tilde{f} := |\nabla h|^{p-2}\Delta_\phi \tilde{f} + (p-2)|\nabla h|^{p-2}\mathrm{Hess}\tilde{f}(\nabla h, \nabla h)$$

with

$$\mathbb{L}_\phi h = \Delta_{p,\phi} h.$$

When $p \neq 2$, the weighted p-Laplacian degenerates or is singular at points $\nabla f = 0$. In this case ε-regularization technique is usually applied by replacing the linearized operator with its approximate operator.

For $\varepsilon > 0$, we define an approximate operator $\mathcal{L}_{\phi,\varepsilon} := \Delta_{p,\phi,\varepsilon}$ for smooth function f_ε by

$$\Delta_{p,\phi,\varepsilon} f_\varepsilon = e^{\phi}\,\mathrm{div}(e^{-\phi} A_\varepsilon^{\frac{p-2}{2}} \nabla f_\varepsilon)$$

with the following characterization

$$\lambda_\varepsilon = \inf_f\{\int_M A_\varepsilon^{\frac{p}{2}} d\mu : \int_M |f_\varepsilon|^p\, d\mu = 1, \int_M |f_\varepsilon|^{p-2} f_\varepsilon\, d\mu = 0\},$$

where $A_\varepsilon = |\nabla f_\varepsilon|^2 + \varepsilon$.

It has been shown in [15] that the infimum above is achieved by a C^∞ eigenfunction f_ε satisfying

$$\Delta_{p,\phi,\varepsilon} f_\varepsilon = -\lambda_\varepsilon |f_\varepsilon|^{p-2} f_\varepsilon$$

with $\lambda_\varepsilon = \int_M A_\varepsilon^{\frac{p-2}{2}} |\nabla f_\varepsilon|^2 d\mu$ by using standard elliptic theory. Taking the limit as $\varepsilon] 0$, we then obtain a continuos weak solution $\lambda_1 = \lim_{\varepsilon] 0} \lambda_\varepsilon$ and $f = \lim_{\varepsilon] 0} f_\varepsilon$.

Define the G_ε norm $\|\cdot\|_{G_\varepsilon}$ for every smooth 2-symmetric tensor V_{ij} by

$$\| V_{ij} \|^2 = (g^{ij} + (p-2)\frac{\nabla_i f_\varepsilon \nabla_j f_\varepsilon}{A_\varepsilon})(g^{kl} + (p-2)\frac{\nabla_k f_\varepsilon \nabla_l f_\varepsilon}{A_\varepsilon}) V_{ik} V_{jl}.$$

Then G_ε trace of V_{ij} is

$$\mathrm{Tr}_{G_\varepsilon}(V_{ij}) = (g^{ij} + (p-2)\frac{\nabla_i f_\varepsilon \nabla_j f_\varepsilon}{A_\varepsilon}) V_{ij}.$$

In particular,

$$\| \mathrm{Hess} f_\varepsilon \|^2 = |\mathrm{Hess} f_\varepsilon|^2 + (p-2)\frac{|\nabla A_\varepsilon|^2}{2A_\varepsilon} + (p-2)^2 \frac{|\langle \nabla A_\varepsilon, \nabla f_\varepsilon\rangle|^2}{A_\varepsilon^2}$$

and

$$\mathrm{Tr}_{G\varepsilon}(\mathrm{Hess} f_\varepsilon) = \Delta f_\varepsilon + (p-2)\frac{\langle \nabla A_\varepsilon, \nabla f_\varepsilon \rangle}{2A_\varepsilon}.$$

Hence,

$$A_\varepsilon^{\frac{p-2}{2}}\, \mathrm{Tr}_{G\varepsilon}(\mathrm{Hess} f_\varepsilon) = \Delta_{p,\varepsilon} f_\varepsilon = \Delta_{p,\phi+\varepsilon} A_\varepsilon^{\frac{p-2}{2}} \langle \nabla\phi, \nabla f_\varepsilon \rangle. \qquad (2.5)$$

3. BEHAVIOUR OF $\lambda_1(t)$ AT THE MAXIMAL TIME

In this section we want to show that the first eigenvalue diverges in a finite time. In [6] Cerbio and Fabricio proved that the first eigenvalue of Laplace-Beltrami operator on a 3-dimensional closed manifold with positive Ricci curvature diverges as $t \to T$ under the Hamilton's Ricci flow. In their work, they used the popular Reilly formula. In [2] the author proved the same result for the weighted Laplacian under Ricci-harmonic flow.

Motivated by [6] and [2], we will show the same result for the eigenvalue of weighted p-Laplacian under the Ricci-harmonic flow. Our derivation will be via weighted p-Reilly formula.

Theorem 3.1 *(Weighted p-Reilly formula [16, Theorem 2.2]). Let $(M, g, d\mu)$ be a compact smooth metric measure space. Then*

$$\int_M (\Delta_{p,\phi} f)^2 - |\nabla f|^{2p-4} \parallel \mathrm{Hess} f \parallel_G^2 d\mu = \int_M |\nabla f|^{2p-4} (Rc + \nabla^2 \phi)(\nabla f, \nabla f) d\mu \qquad (3.1)$$

for $f \in C^\infty(M)$ and

$$\parallel \mathrm{Hess} f \parallel_G^2 = |\mathrm{Hess} f|^2 + \frac{p-2}{2} \frac{|\nabla |\nabla f|^2|^2}{|\nabla f|^2} + \frac{(p-2)^2}{4} \frac{\langle \nabla f, \nabla |\nabla f|^2 \rangle^2}{|\nabla f|^4}$$

Before we state the main result of the section, we remark that it has been proved in [11, Theorem 1.1] that either

$$\limsup_{t \to T} (\max_M R(t)) = \infty \qquad (3.2)$$

or

$$\limsup_{t \to T} (\max_M R(t)) < \infty \quad but \quad \limsup_{t \to T} (\max_M \frac{|W(t)|_{g(t)} + |\nabla^2 \varphi(t)|^2_{g(t)}}{R(t)}) = \infty, \qquad (3.3)$$

where $W(t)$ is the Weyl part of the Riemannian tensor, under the extended Ricci flow for the case $n \geq 3$ and $T < \infty$. Also in this case $|\nabla \varphi|^2$ is uniformly bounded. Observe that if one assumes (3.2) one can easily deduce that

$$\lim_{t \to T} S_{min}(t) = \infty \qquad (3.4)$$

without an additional assumption.

Finally, we use the estimate (3.4) together with (3.1) to prove that the eigenvalues of weighted p-Laplacian diverge as t approaches the maximal time. The main result is the following.

Theorem 3.2 *Let $\lambda_1(t)$ be the first eigenvalue of the weighted p-Laplacian for $p \geq 2$ under the Ricci-harmonic flow $(M, g(t), \phi(t), d\mu)$, $t \in [0, T], T < \infty$ with $S(0) > 0$. Then*

$$\lim_{t \to T} \lambda_1(t) = +\infty, \qquad (3.5)$$

where $S_{ij} - \beta S g_{ij} > 0$ in $M \times [0, T], \beta \in [0, \frac{1}{n}]$.

By the Weighted p-Reilly formula (3.1) we have

$$\int_M (\Delta_{p,\phi} f)^2 - |\nabla f|^{2p-4} \parallel \mathrm{Hess} f \parallel_G^2 d\mu = \int_M |\nabla f|^{2p-4} Sc(\nabla f, \nabla f) d\mu$$
$$+ \int_M |\nabla f|^{2p-4} (\alpha \nabla \phi \otimes \nabla \phi + \nabla^2 \phi)(\nabla f, \nabla f) d\mu. \qquad (3.6)$$

Since $\Delta_{p,\phi} f = \Delta_p f - |\nabla f|^{p-2} \langle \nabla \phi, \nabla f \rangle$, and using an elementary inequality of the form $(a+b)^2 \geq \frac{1}{1+s} a^2 - \frac{1}{s} b^2$ for $s > 0$, we obtain the following inequality

$$\int_M (\Delta_p f)^2 = (\Delta_{p,\phi} f + |\nabla f|^{p-2} \langle \nabla \phi, \nabla f \rangle)^2$$
$$\geq \frac{1}{1+s} (\Delta_{p,\phi} f)^2 - \frac{1}{s} |\nabla f|^{2p-4} |\langle \nabla \phi, \nabla f \rangle|^2. \qquad (3.7)$$

Hence by (2.5) we have

$$|\nabla f|^{2p-4} \parallel \text{Hess}f \parallel_G^2 \geq \frac{1}{n}(|\nabla f|^{p-2}\text{Tr}_G(\text{Hess}f))^2 = \frac{1}{n}(\Delta_p f)^2$$
$$\geq \frac{1}{n(1+s)}(\Delta_{p,\phi} f)^2 - \frac{1}{ns}|\nabla f|^{2p-4}|\langle\nabla\phi,\nabla f\rangle|^2. \quad (3.8)$$

Using (3.8) into the formula below

$$\int_M (\Delta_{p,\phi} f)^2 d\mu = \lambda_1^2 \int_M |f|^{2p-2} d\mu$$

yields

$$\int_M (\Delta_{p,\phi} f)^2 - |\nabla f|^{2p-4} \parallel \text{Hess}f \parallel_G^2 d\mu = (1 - \frac{1}{n(1+s)})\lambda_1^2 \int_M |f|^{2p-2} d\mu$$
$$+ \frac{1}{ns}|\nabla f|^{2p-4}|\langle\nabla\phi,\nabla f\rangle|^2. \quad (3.9)$$

Putting (3.9) into (3.6) gives

$$(1 - \frac{1}{n(1+s)})\lambda_1^2 \int_M |f|^{2p-2} d\mu + \frac{1}{ns}\int_M |\nabla f|^{2p-4}|\langle\nabla\phi,\nabla f\rangle|^2 d\mu$$
$$\geq \int_M |\nabla f|^{2p-4} Sc(\nabla f,\nabla f) d\mu + \alpha \int_M |\nabla f|^{2p-4}\nabla\phi\otimes\nabla\phi(\nabla f,\nabla f)\, d\mu \quad (3.10)$$
$$+ \int_M |\nabla f|^{2p-4}\nabla^2\phi(\nabla f,\nabla f)\, d\mu.$$

Choosing $s := \frac{1}{\alpha_n}, \alpha \geq \alpha_n > 0$, we have

$$1 - \frac{1}{n(1+s)} = \frac{n(1+\alpha_n)-\alpha_n}{n(1+\alpha_n)} \quad \text{and} \quad \frac{1}{ns} = \frac{\alpha_n}{n}$$

we observe that

$$\alpha \int_M |\nabla f|^{2p-4}\nabla\phi\otimes\nabla\phi(\nabla f,\nabla f)\, d\mu \geq \frac{\alpha_n}{n}\int_M |\nabla f|^{2p-4}|\langle\nabla\phi,\nabla f\rangle|^2 d\mu$$

by identifying $\nabla\phi\otimes\nabla\phi(\nabla f,\nabla f)$ with $|\langle\nabla\phi,\nabla f\rangle|^2$. Hence (3.10) reads

$$(\frac{n(1+\alpha_n)-\alpha_n}{n(1+\alpha_n)})\lambda_1^2 \int_M |f|^{2p-2} d\mu \geq \int_M |\nabla f|^{2p-4} Sc(\nabla f,\nabla f) d\mu$$
$$+ \int_M |\nabla f|^{2p-4}\nabla^2\phi(\nabla f,\nabla f) d\mu. \quad (3.11)$$

Since ϕ solves the heat equation we observe that $|\nabla^2\phi| \geq \frac{1}{\sqrt{n}}|\Delta\phi| = \frac{1}{\sqrt{n}}|\phi_t|$. Using the condition $S_{ij} - \beta S g \geq 0$, then (3.11) implies

$$(\frac{n(1+\alpha_n)-\alpha_n}{n(1+\alpha_n)})\lambda_1^2 \int_M |f|^{2p-2} d\mu \geq \beta \int_M S|\nabla f|^{2p-2} d\mu \frac{1}{\sqrt{n}}\min_M|\phi_t| \int_M |\nabla f|^{2p-2} d\mu$$
$$\geq (\beta S_{min}(t) + \frac{1}{\sqrt{n}}\min_M|\phi_t|) \int_M |\nabla f|^{2p-2} d\mu. \quad (3.12)$$

Multiplying both sides of (2.2) by the quantity $|f|^{p-2}f$ and integrate over M using integration by parts formulas to arrive at

$$\lambda_1 \int_M |\nabla f|^{2p-2} d\mu = (p-1)\int_M |\nabla f|^p |f|^{p-2} d\mu.$$

Applying the Hölder inequality for any $p > 2$, we have

$$\lambda_1 \int_M |f|^{2p-2} d\mu \leq (p-1)(\int_M |\nabla f|^{2p-2} d\mu)^{\frac{p}{2p-2}}(\int_M |f|^{2p-2})^{\frac{p-2}{2p-2}},$$

hence

$$\int_M |\nabla f|^{2p-2} d\mu \geq \left(\frac{\lambda_1}{p-1}\right)^{\frac{2p-2}{p}} \int_M |f|^{2p-2} d\mu \tag{3.13}$$

Putting (3.13) into (3.12) we arrive at

$$\left(\frac{n(1+\alpha_n)-\alpha_n}{n(1+\alpha_n)}\right)\lambda_1^2 \int_M |f|^{2p-2} d\mu$$
$$\geq \left(\beta S_{min}(t) + \frac{1}{\sqrt{n}} \min_M |\phi_t|\right)\left(\frac{\lambda_1}{p-1}\right)^{\frac{2p-2}{p}} \int_M |f|^{2p-2} d\mu. \tag{3.14}$$

For $p > 2$, we can conclude that

$$\left(\frac{n(1+\alpha_n)-\alpha_n}{n(1+\alpha_n)}\right)\lambda_1^2 \geq \left(\beta S_{min}(t) + \frac{1}{\sqrt{n}} \min_M |\phi_t|\right)\left(\frac{\lambda_1}{p-1}\right)^{\frac{2p-2}{p}} \tag{3.15}$$

and then

$$\left(\frac{n(1+\alpha_n)-\alpha_n}{n(1+\alpha_n)}\right)\lambda_1^{\frac{2}{p}} \geq \left(\beta S_{min}(t) + \frac{1}{\sqrt{n}} \min_M |\phi_t|\right)\left(\frac{1}{p-1}\right)^{\frac{2p-2}{p}} \tag{3.16}$$

which finally implies

$$\lambda_1(t) \geq \left[\frac{n(1+\alpha_n)}{n(1+\alpha_n)-\alpha_n}\left(\beta S_{min}(t) + \frac{1}{\sqrt{n}} \min_M |\phi_t|\right)\right]^{\frac{p}{2}} \cdot (p-1)^{1-p}. \tag{3.17}$$

Since $S_{min}(t) \to +\infty$ as $t \to T$ and $\min_M |\phi_t|$ is finite then $\lim_{t \to T} = +\infty$. This completes the proof of the theorem.

Remark 3.3 *The above result also holds for the case $p = 2$. Indeed (3.14) reduces to*

$$\left(\frac{n(1+\alpha_n)-\alpha_n}{n(1+\alpha_n)}\right)\lambda_1^2 \int_M |f|^2 d\mu \geq \left(\beta S_{min}(t) + \frac{1}{\sqrt{n}} \min_M |\phi_t|\right)\left(\frac{\lambda_1}{p-1}\right) \int_M |f|^2 d\mu. \tag{3.18}$$

for $p = 2$ and consequently,

$$\lambda_1(t) \geq \frac{n(1+\alpha_n)}{n(1+\alpha_n)-\alpha_n}\left(\beta S_{min}(t) + \frac{1}{\sqrt{n}} \min_M |\phi_t|\right). \tag{3.19}$$

Then Theorem 3.2 reduces to [2, Theorem 2.5].

4. SUMMARY

In this paper, we investigated the behaviour of the first non-zero eigenvalue of the weighted p-Laplacian acting on the space of smooth functions along an abstract geometric flow, called the Ricci-harmonic flow. Note that the first eigenvalue's properties determine to large extent the behaviour of the spectrum of a self-adjoint operator. We have shown that the first eigenvalue diverges in maximal existence time for the flow. Our derivation was done via weighted p-Reilly formula. This property is a powerful tool in the study of geometry and topology of the underlying space. This result also generalizes a number of results. Other properties of the first eigenvalue, such as monotonicity, differentiability and asymptoticity, will also be discussed somewhere else.

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