Article Citation Format
Skwame, Y. & Adeyeye, F.J. (2019): The A-stable of the 5th Higher Order Of Hybrid Block Simpson's Method For Stiff ODEs.
Journal of Advances in Mathematical & Computational Sciences
Vol. 7, No. 1. Pp 51-60

Article Progress Time Stamps

Article Type: Research Article
Manuscript Received 17th Jan. 2019
Review Type: Blind
Final Acceptance: 27th March, 2019
Article DOI: dx.doi.org/10.22624/AIMS/MATHS/V7N1P7

The A-stable of the 5th Higher Order Of Hybrid Block Simpson's Method For Stiff ODEs

[1]**Skwame, Y.** & [2]**Adeyeye, F.J.**
[1]Department of Mathematical sciences, Adamawa State University, Mubi, Nigeria
[2]Department of Math/Computer Science, Federal University of Petroleum Resources, Effurun, Nigeria
Corresponding E-mail: adeyeye.fola@fupre.edu.ng

ABSTRACT

The generation of a self-starting Simpson's type block hybrid method (BHM) consisting of very closely accurate members each of order p=q+2 as a block is constructed. We got the higher order members of each by increasing the number k in the multi-step collocation (MC) used to derive the k-step continuous formula (5) through the aid of MAPLE software program. This paper presents the generation of hybrid block schemes (CHBS) through the addition of one off-mesh collocation points in the MC. The (CHBS) is evaluated along with its first derivative where necessary to give a hybrid block schemes for a simultaneous application to the ordinary differential equations (ODEs}. Some test problems to confirm the reliability of these scheme of experiments were carried out.

Keywords: Continuous hybrid block schemes (CHBS), Multi-step collocation (MC), hybrid block schemes, ODEs. the k-step continuous formula.

1. INTRODUCTION

Accuracy, Stability and efficiency are the three conflicting basic aims in the design of schemes to solve ordinary differential equations. In [1,2,3] we identify a continuous hybrid block scheme (CHBS) through the addition of one or more off-mesh collocation points in the multi-step collocation (MC) as represented by (2.1.8). The (CHBS) is evaluated at some distinct points involving mesh and off-mesh points along with its first derivative, where necessary, to give multiple hybrid block schemes for the treatment of stiff ordinary differential equations. This paper is classified into sections. Section 1.0 is definitions of terms, we restate the MC procedure involving off-mesh collocation points for each k and we analyze on its convergence analysis obtained in a block form in section 2.0, by obtaining the order and error constants in a block form, the stability regions are also plotted. Section 3.0 is the numerical implementation of the block hybrid schemes on stiff (ODEs) and we give conclusion in section 4.0.

Definition 1.1 Linear Multi-Step Method
A k-step linear multi-step (LMM) is of the form

$$\sum_{j=0}^{k}\alpha_j y_{n+j} = h\sum_{j=0}^{k}\beta_j f_{n+j} \qquad (1.1)$$

Where

$$f_{n+j} = f(x_{n+j}, y_{n+j}), \qquad y_{n+j} = y(x_{n+j})$$

α_j and β_j are constants and satisfy the constraints

$$\alpha_k \neq 0, \alpha_0^2 + \beta_0^2 > 0$$

(1.1) is explicit if $\beta_k = 0$ and implicit if $\beta_k \neq 0$

2. DERIVATION TECHNIQUES OF MC.

Let us consider the first order system of ODEs

$$y^1 = f(x,y), \qquad a < x < b, \quad y,f \in \Re^s \qquad (2.1)$$

where y satisfies a given set of s associated conditions, which are either all initial, all boundary or mixed conditions. The idea of the k – step MC, following [4], is to find a polynomial U of the form

$$U(x) = \sum_{j=0}^{t-1}\varphi_j(x)y_{n+j} + h\sum_{j=0}^{m-1}\phi_j(x)f\left(x_j, u(\overline{x}_j)\right), x_n \leq x \leq x_{n+x} \rightleftarrows \qquad (2.2)$$

Where t denotes the number of interpolation points $x_{n+i}, i = 0,1,..,t-1,$ and m denote the number of distinct collocation points $\overline{x_i} \in [x_n, \ x_{n+k}], \ i = 0,1,---, m-1$ *the points* $\overline{x_i}$ are chosen from the step x_{n+i} as well as one or more off step points.

The following assumptions are made;

1. Although the step size can be variable, for simplicity in our presentation of the analysis in this paper, we assume it is constant $h = x_{n+1} - x_n, \quad N = \frac{b-a}{h}$ with the steps given by
$\{x_n / x_n = a + nh, \ n = 0,1,---, N\},$

2. That (2.1) has a unique solution and the coefficients $\phi_j(x)$, $\varphi_j(x)$ in (2.2) can be represented by polynomials of the form

$$\varphi_j(x) = \sum_{i=0}^{t+m-1} \varphi_{j,i+1} x^i, j\varepsilon\{0,1,---,t-1\} \tag{2.3}$$

$$h\phi_j(x) = \sum_{i=0}^{t+m-1} \phi_{j,i+1} x^i j\varepsilon\{0,1,---,m-1\} \tag{2.4}$$

with constant coefficients $\phi_{j,i+1}$, $h\varphi_{j,i+1}$ to be determined using the interpolation and collocation conditions:

$$u(x_{n+i}) = y_{n+i}\ , i\varepsilon\ \{0,1,---,t-1\} \tag{2.5}$$

$$u^1(\overline{x}_i) = f\left(\overline{x}_i, u(\overline{x}_j)\right), j\varepsilon\{0,1,---,m-1\} \tag{2.6}$$

With this assumptions we obtain an MC polynomial, following [4, 5], in the form

$$u(x) = \sum_{i=0}^{t+m-1} a_i x^i, a^i = \sum_{j=0}^{t-1} c_{i+1,j+1} + \sum_{j=0}^{m-1} c_{i+1,j+t+1} f_{n+j} \tag{2.7}$$

Where $x_n \le x \le x_{n+k}$ and $c_{ij}\ i, j\varepsilon\{1,2,---,t+m\}$ are constants given by the elements of the inverse matrix $C = D^{-1}$. The MC matrix D is a nonsingular (m+1) square matrix of the type

$$D = \begin{bmatrix} 1 & x_n & x_n^2 & - & - & - & x_n^{t+m-1} \\ 1 & x_{n+1} & x_{n+1}^2 & - & - & - & x_{n+1}^{t+m-1} \\ . & . & . & . & & & . \\ . & . & . & & . & & . \\ . & . & . & & & . & . \\ 1 & x_{n+t-1} & x_{n+t-1}^2 & - & - & - & x_{n+t-1}^{t+m-1} \\ 0 & 1 & 2\overline{x_0} & - & - & - & (t+m-1)\overline{x_0^{t+m-1}} \\ 0 & 1 & 2\overline{x_1} & - & - & - & (t+m-1)\overline{x_1^{t+m-1}} \\ . & . & . & . & & & . \\ . & . & . & & . & & . \\ . & . & . & & & . & . \\ 0 & 1 & 2\overline{x_{m-1}} & - & - & - & (t+m-1)\overline{x_{m-1}^{t+m-1}} \end{bmatrix} \tag{2.8}$$

2.1 Five steps Block Hybrid Simpson's Method with one off-step point.

The parameters required for equation (2.8) are k=5, t=1, m= k+2; $\left(x_n\ , x_{n+1}\right)$;

$$\left(\bar{x}_0 = x_n\ ,\ \bar{x}_1 = x_{n+1}\ ,\ \bar{x}_2 = x_{n+2}\ ,\ \bar{x}_3 = x_{n+3}\ ,\ \bar{x}_4 = x_{n+4}\ ,\ \bar{x}_{\frac{9}{2}} = x_{n+\frac{9}{2}}\ ,\ \bar{x}_5 = x_{n+5}\right)$$

Hence the matrix (2.8) takes the following shape.

$$D = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 & 6x_n^5 & 7x_n^6 \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 & 5x_{n+1}^4 & 6x_{n+1}^5 & 7x_{n+1}^6 \\ 0 & 1 & 2x_{n+2} & 3x_{n+2}^2 & 4x_{n+2}^3 & 5x_{n+2}^4 & 6x_{n+2}^5 & 7x_{n+2}^6 \\ 0 & 1 & 2x_{n+3} & 3x_{n+3}^2 & 4x_{n+3}^3 & 5x_{n+3}^4 & 6x_{n+3}^5 & 7x_{n+3}^6 \\ 0 & 1 & 2x_{n+4} & 3x_{n+4}^2 & 4x_{n+4}^3 & 5x_{n+4}^4 & 6x_{n+4}^5 & 7x_{n+4}^6 \\ 0 & 1 & 2x_{n+\frac{9}{2}} & 3x_{n+\frac{9}{2}}^2 & 4x_{n+\frac{9}{2}}^3 & 5x_{n+\frac{9}{2}}^4 & 6x_{n+\frac{9}{2}}^5 & 7x_{n+\frac{9}{2}}^6 \\ 0 & 1 & 2x_{n+5} & 3x_{n+5}^2 & 4x_{n+5}^3 & 5x_{n+5}^4 & 6x_{n+5}^5 & 7x_{n+5}^6 \end{bmatrix} \qquad (2.9)$$

Using the maple software environment to evaluate (2.9) at the grid points

$x = x_{n+1}\ , x = x_{n+2}\ , x = x_{n+3}\ , x = x_{n+4}\ ,\ x = x_{n+\frac{9}{2}}\ ,\ x = x_{n+5}$

We obtain the six discrete schemes, namely,

$$y_{n+1} = y_n + h\left[\frac{28199}{90720}f_n + \frac{78553}{70560}f_{n+1} - \frac{4519}{5040}f_{n+2} + \frac{13691}{15120}f_{n+3} - \frac{9841}{10080}f_{n+4} + \frac{13808}{19845}f_{n+\frac{9}{2}} - \frac{1537}{10080}f_{n+5}\right]$$

$$y_{n+2} = y_n + h\left[\frac{169}{567}f_n + \frac{6691}{4410}f_{n+1} - \frac{5}{63}f_{n+2} + \frac{517}{945}f_{n+3} - \frac{206}{315}f_{n+4} + \frac{9472}{19845}f_{n+\frac{9}{2}} - \frac{67}{630}f_{n+5}\right]$$

$$y_{n+3} = y_n + h\left[\frac{1013}{3360}f_n + \frac{11601}{7840}f_{n+1} + \frac{45}{112}f_{n+2} + \frac{689}{560}f_{n+3} - \frac{1017}{1120}f_{n+4} + \frac{464}{735}f_{n+\frac{9}{2}} - \frac{153}{1120}f_{n+5}\right]$$

$$y_{n+4} = y_n + h\left[\frac{170}{567}f_n + \frac{3296}{2205}f_{n+1} + \frac{104}{315}f_{n+2} + \frac{1664}{945}f_{n+3} - \frac{63}{315}f_{n+4} + \frac{8192}{19845}f_{n+\frac{9}{2}} - \frac{32}{315}f_{n+5}\right]$$

$$y_{n+\frac{9}{2}} = y_n + h\left[\frac{2151}{7168}f_n + \frac{374463}{250880}f_{n+1} + \frac{1215}{3584}f_{n+2} + \frac{31077}{17920}f_{n+3} + \frac{3159}{35840}f_{n+4} + \frac{162}{245}f_{n+\frac{9}{2}} - \frac{4131}{35840}f_{n+5}\right]$$

$$y_{n+5} = y_n + h\left[\frac{5435}{18144}f_n + \frac{21125}{14112}f_{n+1} + \frac{325}{1008}f_{n+2} + \frac{5375}{3024}f_{n+3} - \frac{125}{2016}f_{n+4} + \frac{4400}{3969}f_{n+\frac{9}{2}} + \frac{115}{2016}f_{n+5}\right]$$

....2.2.0

2.2 The Order and Error constants of the A-stable Block Hybrid Methods.
The hybrid block methods which are obtained in a block form with the help of maple software have the following order and error constants for each case.

Case k=5

Evaluating points	*Order*	*Error Constants*
$y(x = x_{n+1})$	7	$\frac{1759}{211680}$
$y(x = x_{n+2})$	7	$\frac{337}{52920}$
$y(x = x_{n+3})$	7	$\frac{57}{7840}$
$y(x = x_{n+4})$	7	$\frac{44}{6615}$
$y\left(x = x_{n+\frac{9}{2}}\right)$	7	$\frac{54351}{8028160}$
$y(x = x_{n+5})$	7	$\frac{275}{42336}$

Table 1: k=5 **BHSM** with one off-step point

The method k=5 is of order 7 as a block and has error constants

$$C_8 = \left(\frac{1759}{211680}, \frac{337}{52920}, \frac{57}{7840}, \frac{44}{6615}, \frac{54351}{8028160}, \frac{275}{42336}\right)^T$$

2.3 Stability Regions of the Block Hybrid Simpson's Methods.

To compute and plot the absolute stability regions of the block hybrid Simpson's methods, the methods are reformulated as general linear methods expressed as;[4,5,6]

$$\begin{bmatrix} Y \\ -- \\ y_{i+1} \end{bmatrix} = \begin{bmatrix} A & / & U \\ - & - & - \\ B & / & V \end{bmatrix} \begin{bmatrix} hf(Y) \\ \\ y_{i-1} \end{bmatrix}$$

where,

$$A = \begin{bmatrix} a_{11} & . & . & . & a_{1s} \\ . & . & & & . \\ . & & . & & . \\ . & & & . & . \\ a_{s1} & . & . & . & a_{ss} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & . & . & . & b_{1s} \\ . & . & & & . \\ . & & . & & . \\ . & & & . & . \\ b_{k1} & . & . & . & b_{ks} \end{bmatrix}$$

$$Y = \begin{bmatrix} y_n \\ y_{n+1} \\ . \\ . \\ . \\ y_{n+k} \end{bmatrix} \qquad y_{i+1} = \begin{bmatrix} y_{n+k} \\ . \\ . \\ y_{n+k-1} \end{bmatrix} \qquad y_{i-1} = \begin{bmatrix} y_{n+k-1} \\ . \\ . \\ y_{n+k-2} \end{bmatrix}$$

and the elements of the matrices A,B,U and V are obtained from the interpolation and collocation and collocation points.

The elements of the matrices A, B, U and V are substituted into the stability matrix

$$M(z) = B_2 + zA_2(I - zA_1)^{-1}B_1 \qquad where \;\; A_1 = A, A_2 = B, B_1 = U, B_2 = V$$

and the stability function

$$\rho(\eta, z) = \det(\eta I - M(z))$$

Computing the stability function with Maple yields the stability polynomial of the method which is plotted in Matlab to produce the required absolute stability region of the method.

2.4 Absolute stability region of the block hybrid method K=5 [4,5,6,7]]

The block hybrid methods (2.1) with one off-grid point are arranged as shown below;
The coefficients of these methods expressed in tabular form bellow gives the coefficients of the new method.

$$\begin{bmatrix} y_n \\ y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y_{n+\frac{9}{2}} \\ y_{n+5} \\ --- \\ y_{n+5} \\ y_{n+4} \\ y_{n+3} \\ y_{n+2} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & 0 & 0 & 0 & 0 & 1 \\ \frac{28199}{90720} & \frac{78553}{70560} & -\frac{4519}{5040} & \frac{13691}{15120} & -\frac{9841}{10080} & \frac{13808}{19845} & -\frac{1537}{10080} & . & 0 & 0 & 0 & 0 & 1 \\ \frac{169}{567} & \frac{6691}{4410} & -\frac{5}{63} & \frac{517}{945} & -\frac{206}{315} & \frac{9472}{19845} & -\frac{67}{630} & . & 0 & 0 & 0 & 0 & 1 \\ \frac{1013}{3360} & \frac{11601}{7840} & \frac{45}{112} & \frac{689}{560} & -\frac{1017}{1120} & \frac{464}{735} & -\frac{153}{1120} & . & 0 & 0 & 0 & 0 & 1 \\ \frac{170}{567} & \frac{3296}{2205} & \frac{104}{315} & \frac{1664}{945} & -\frac{62}{315} & \frac{8192}{19845} & -\frac{32}{315} & . & 0 & 0 & 0 & 0 & 1 \\ \frac{2151}{7168} & \frac{374463}{250880} & \frac{1215}{3584} & \frac{31077}{17920} & \frac{3159}{35840} & \frac{162}{245} & -\frac{4131}{35840} & . & 0 & 0 & 0 & 0 & 1 \\ \frac{5435}{18144} & \frac{21125}{14112} & \frac{325}{1008} & \frac{5375}{3024} & -\frac{125}{2016} & \frac{4400}{3969} & \frac{115}{2016} & . & 0 & 0 & 0 & 0 & 1 \\ - & - & - & - & - & - & - & - & - & - & - & - & - \\ \frac{5435}{1814} & \frac{21125}{14112} & \frac{325}{1008} & \frac{5375}{3024} & -\frac{125}{2016} & \frac{4400}{3969} & \frac{115}{2016} & . & 0 & 0 & 0 & 0 & 1 \\ \frac{170}{567} & \frac{3296}{2205} & \frac{104}{315} & \frac{1664}{945} & -\frac{62}{315} & \frac{8192}{19845} & -\frac{32}{315} & . & 0 & 0 & 0 & 0 & 1 \\ \frac{1013}{3360} & \frac{11601}{7840} & \frac{45}{112} & \frac{689}{560} & -\frac{1017}{1120} & \frac{464}{735} & -\frac{153}{1120} & . & 0 & 0 & 0 & 0 & 1 \\ \frac{169}{567} & \frac{6691}{4410} & -\frac{5}{63} & \frac{517}{945} & -\frac{206}{315} & \frac{9472}{19845} & -\frac{67}{630} & . & 0 & 0 & 0 & 0 & 1 \\ \frac{28199}{90720} & \frac{78553}{70560} & -\frac{4519}{5040} & \frac{13691}{15120} & -\frac{9841}{10080} & \frac{13808}{19845} & -\frac{1537}{10080} & . & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+\frac{9}{2}} \\ f_{n+5} \\ - \\ y_{n+4} \\ y_{n+3} \\ y_{n+2} \\ y_{n+1} \\ y_n \end{bmatrix}$$

Substituting the values of A, B, U and V into the stability matrix and the stability function and using Maple software yields the stability polynomial of the block method.

Using a Matlab program, we obtained the stability region of the block hybrid Simpson's method for K= 5 as shown in figure 1. The stability region of the block method shows that it is A-stable.

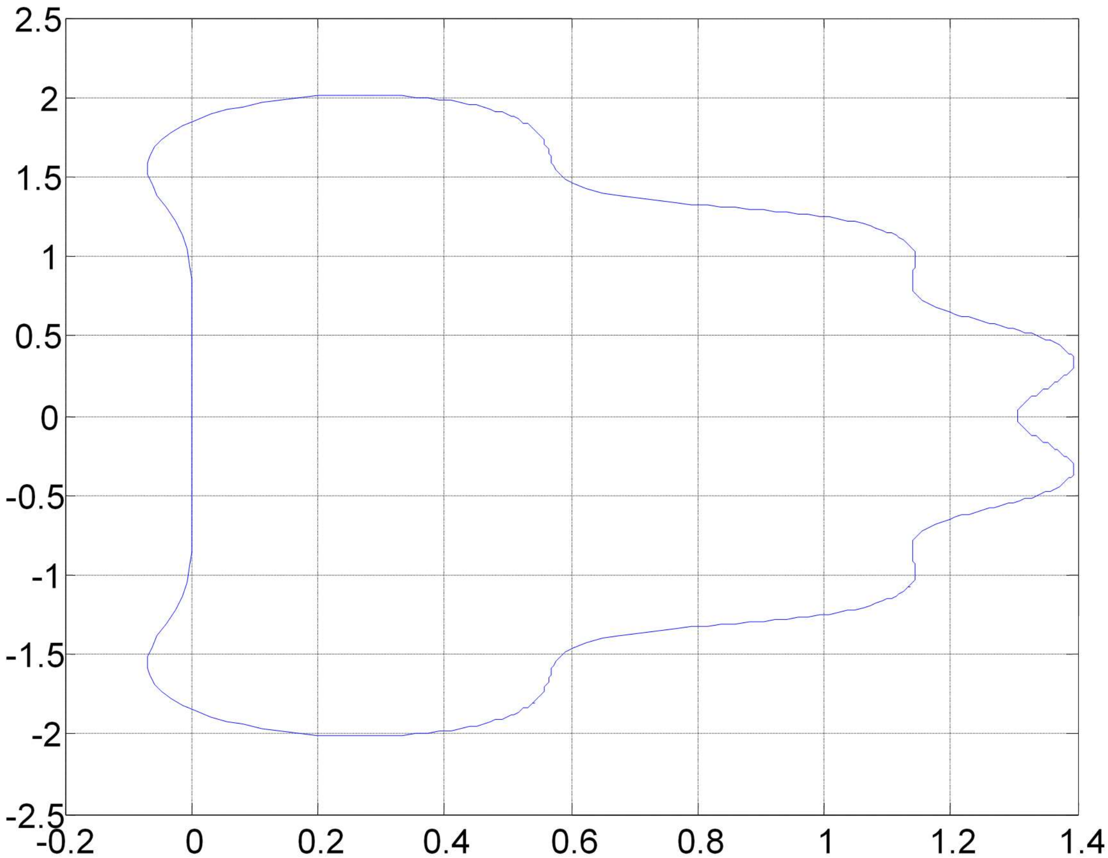

Figure 1: Stability region of the block hybrid Simpson's method K=5

3.0 NUMERICAL IMPLEMENTATION.

To study the efficiency [8,9,10,11] of the block hybrid method for $2 \leq k \leq 7$, we present some numerical examples widely used by several authors such as [8,9,10,11].

$$Experiment 1 \quad y^1 = -1000000y, \quad where\ h = 0.1, \quad x\varepsilon[0,1.8]$$

$$Exact\, solution \quad y(x) = e^{-1000000x}$$

Table of absolute errors for experiment 1

Y	A-stable Hybrid Block Simpson's K=5
0.1	1.56×10^{-1}
0.2	5.56×10^{-2}
0.3	3.33×10^{-2}
0.4	2.22×10^{-2}
0.5	1.11×10^{-1}
0.6	1.73×10^{-2}
0.7	6.17×10^{-3}
0.8	3.70×10^{-3}
0.9	2.47×10^{-3}
1.0	1.23×10^{-2}
1.1	1.92×10^{-3}
1.2	6.86×10^{-4}
1.3	4.11×10^{-4}
1.4	2.74×10^{-4}
1.5	1.37×10^{-3}
1.6	2.13×10^{-4}
1.7	7.62×10^{-5}
1.8	4.57×10^{-5}

Absolute Errors Of A-Stable New BHSM AT X $\in [0, 1.8]$ $\lambda = -1000000$

4. CONCLUSION

The generation of the 5th hybrid block scheme **(CHBS)**, the A - Stability, Convergence , Accuracy from the stability graph, the numerical table results is evident that the proposed schemes is indeed numerically viable and can handle stiff equations. The result on the Numerical table shows the strength of the 5th block scheme.
It therefore implies that if a further block is generated better rate of convergence will be achieved

REFERENCES

[1] G. Dahlquist ." A special stability problem for linear Multistep methods" BIT3 27-43.

[2] P. Onumanyi, , D. O. Awoyemi S. N. Jator, and U. W. Sirisena, " New linear multi-step methods with continuous coefficients For first order initial value problems". J. Nig. Math. Soc. 13, 37-51, 1994.

[3] J. D. Lambert, ."Numerical methods for ordinary differential Systems ,Wiley" p220,1991.

[4] D.G. Yakubu," New block implicit Runge-Kuta Collocation methods for solving stiff differential equations.Mathematical sciences programme", Abubakar Tafawa Balewa University, Bauchi, Nigeria.

[5] R. England and, R. M. Mattheij." Sequential step control for Integration of two-point boundary value problems" . Department of Mathematics, Catholic University,Toemooiveld, 6255 Report 8541, 1985.

[6] Y. Skwame." Construction of Simpson-Type Hybrid Block methods for Stiff ordinary differential equations."M. Sc. Project. (Unpublished). University of Jos, Jos, Nigeria,2005.

[7] J. P. A Chollon,"Study of block hybrid Adams methods with link To two step Runge-Kutta methods for first order ordinary differentia Equations". Ph. D. Thesis. (2004). University of Jos, Jos, Nigeria.

[8] Adeyeye FJ (2015). The solution of some initial value problems in ordinary differential equation (ODE) by some linear Multistep method(LMM). African journals of Computer and ICT IEEE Vol 8 Issue 2 pp 31-38 Foreign

[9] M. Nakashima," Variable coefficients A-Stable explicit Runge-Kutta methods". Proceeding of the fifth international conference on scientific computing. University of Benin, Benin City, Nigeria; Pp 138-143.1992.

[10] Adeyeye FJ, Igobi D.K, Ibijola E.A.(2015): A news Hybrid in the nonlinear part of ADM for initial Value problem of ordinary Differential Equation. Journal of Mathematical Research, Application (7) pp 102-109

[11] E.A. Ibijola, , Y. , Skwame, G. Kumleng." Formation of Hybrid Block Method of Higher Step-sizes, through the continuous multi-step collocation", 2(2), pp. 161- 173. 2011.